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RELATIVITY AND ITS PHILOSOPHIC IMPLICATIONS.*

Aeltestes bewahrt mit Treue,
Freundlich aufgefasstes Neue.
—Goethe.

OF all the natural sciences physiology would seem to touch the common interests of mankind most broadly and closely. As it becomes more and more rigorously scientific, it tends to resolve itself into chemistry. Witness the recent experiments of Loeb and Northrop on the aseptic flies of Bogolanow and Guyénot, indicating that the natural duration of an organism's life is the time necessary to complete a chemical reaction or a series of such reactions, and that it is doubled or trebled by lowering the temperature 10° C. Similarly, as chemistry becomes more strictly scientific, it tends to resolve into physics, and physics tends likewise to pass over into mechanics, which, we all know, lapses into mathematics. Mechanics has been defined by Kirchhoff, with startling clearness and exactness, thus: "Mechanics is the science of motion. Its problem is to describe the motions that take place in nature, to describe them completely and in the simplest way." A description is complete when it answers every rational question that can be put, but to say when it is simplest is not so easy. The simplest description possible to-day may not be the simplest possible to-morrow. Anyway, the final problem of all the physical sciences is thus seen to be *the description of the motions of the physical universe.*

*Address delivered at the New Orleans meeting of the Southern Society of Psychology and Philosophy, April, 1920.

Motion is change of place, or position in space, during time. To describe it is to tell how position changes as time goes on. Any point or position in our space is determined and distinguished from every other by three mutually independent measures expressed by numbers. Thus, if you know how far, that is how *many* feet, the point of this pencil is from the end-wall, from the side wall, and from the floor, then you know precisely where it is and where no other point is. Similarly, to know where a star is, you must know just three such things: as, its angular distances from due east and from the horizon—these two will enable you to point at it—and lastly, its distance in the direction thus determined. Three such things you *must know,* and anything else independent of these three you *cannot* know, about a position in our space. Since *three* such measurements are necessary and enough to fix any point in our space, we say that space is *three-dimensional.* Any three such measurements for a point are called its *coördinates.* We are all familiar with such, as latitude and longitude, which *two* fix a position on the earth-surface, which surface is accordingly called *two*-dimensional. We use such coördinates in addressing a letter, as 74 Ninth Street. Here 74 and 9 are the two coördinates fixing the place on the surface. If further we specify tenth floor, the 10 is the third coördinate, fixing the position in three-dimensional space.

All sorts of coördinates are possible and in actual use. The simplest are the so-called rectangular Cartesian (from their inventor, Descartes), the first already mentioned, as the three distances of a point from three walls or planes concurrent in a point called the origin, the starting-point, and denoted by the letter O. These three coördinate lengths are conveniently measured on the three edges of the room, meeting in the origin O; these edges are straight, lines at right angles, each to the other two, and are called the rectangular axes of X, Y, Z; the three lengths or co-

ördinates are also denoted by ex, y, z; thus, the pencil-point, e. g., may be the point (5, 9, 12). Every such set of three values represents a point, and conversely, every point represents a set of three such values.

Each of us carries a set of such axes in his own body; the up-and-down axis, the right-and-left axis—formed by the out-stretched arms, and the fore-and-aft axis (a horizontal line, which we may imagine passing directly through the breast and back, midway of the shoulders). We tell where a point is by telling how high or low it is, how far to the right—or left, how far to the front or behind, up, right and front being marked +, down, left, and behind marked —. As we move hither and thither, we change the position of these axes, and this change expressed in mathematical terms is called *transformation of coördinates.* Any point whatever may serve as Origin O, and in space there is a threefold infinity of such possible origins; every hither-thither motion is a change of origin. A man may also wheel round, remaining upright; here the origin remains fixed, but the two horizontal axes are rotated round the third or vertical axis. Similarly he may rotate on a horizontal bar round the right-and-left axis, or even by a special mechanical contrivance round the fore-and-aft axis, and in each case through any angle whatever. Here again there is then a threefold infinity of possible changes of axes, making in all a six-fold infinity of possible transformations of coördinates. Any single transformation involving both any change of origin and any rotation of axis may be expressed by a set of three simple equations. The totality of all such possible transformation is called a *Group*, that is an aggregate of operations, such that the product of any two is itself some third member of the Group.

The stress here laid upon these notions has been intentional though reluctant; for they cannot be stressed too

heavily. Indeed, the very heart and soul of the New Mechanics is this same *Transformation of Coördinates.*

Passing now to our subject-matter proper, we note that all men are familiar with the fact that motion is in some sense relative. You sit motionless in the car while it speeds along the track. You move with it but not in it; you are at rest relatively to the car but in motion relatively to the track and all outside space. A child knows all this but does not estimate the facts at their full significance. To bring them out clearly, let us imagine a number of coëxistent compunctual spaces. This need not bewilder anyone. You may take any rule and slip its edge along another rule-edge or any straight line. Here one straight line coincides with and slides along another. Now consider a surface. Suppose you pull on a perfect-fitting glove. Here the inside surface fits and moves on the surface of your hand; the two surfaces coincide point for point, and one moves on the other. Still better, since these surfaces are not isotropic, suppose a perfect sphere covered with a delicate film or gauze; then we may imagine this later slipping round over the sphere surface every way, the two always coincident point for point. Quite similarly we may imagine one isotropic space coinciding with another point for point and yet moving anyway in that other, always maintaining exact coincidence. So, too, we may imagine a third space moving in the second, a fourth in the third, and so on. We may imagine each of these spaces with its own proper color, red or green, blue or yellow, and all coexistent like all the colors in the white sunbeam. Or we may imagine a single point, when it moves, as staining each space with its track of a different color in each, red in one, green in another, and so on; and this seems to be very important.

Now imagine a man on a ship sailing west on the Equator 20^m an hour. He looks over the ship's side and drops a penny into the deep. It falls down straight, as he

sees it—that is, it makes (say) a straight red track in the man's space, Sm. But not in the ocean space, So. To a man out there at rest on an island the penny falls to the sea on a green path, a parabola. Or consider simply the man. In his own ship-space, Sm, he is at rest. But in the Equator space, Sq, he is moving on a green path parallel to the Equator 20^m an hour west. Meantime, this Equator space is itself whirling eastward round the earth's axis in an axial space, Sx, at 1000^m an hour. In this Sx, then, the man is making a circular blue streak 980 miles an hour. Meantime this axis is being swept always parallel to itself round the sun on an immense ellipse at 18.6 miles per second. In this sun-space, Ss, then the man is tracing an enormous yellow spiral, one spire per day, moving round it 980 miles an hour and forward on it 18.6 miles a second. Meantime, the sun itself with its space is moving through universal space, Su, at a great rate, and in this Su the man is tracing out a purple path of almost indescribable complexity. Each and every one of these paths is just as actual as any other, each is relative to a higher space supposed at rest relative to the other space moving in it.

The question now arises, when does this stop? Where shall we find a space that is at rest absolutely, that is not swept along in some still higher space, so that with respect to this space any path will be the true path and any motion the true motion? You see at once that this sounds a little like asking for the true beginning of time and of the world. However, Newton and his successors believed firmly in an absolute space and an absolute time, though admitting of course the relativity of all observations so far. But could such an absolute space be actually found? The supposed answer was, Yes! For was not such an absolute actually present in the (assumed) universal aether which was (supposed to be) absolutely at rest? It was indeed the long and strenuous contention of George Gabriel Stokes that the

ether was carried along by the planets in motion, much as the air is carried in a closed car; but profound researches especially of Lorentz, the great continuator of Maxwell, showed that this could not be, that electro-magnetic-optical measurements required that the aether be stationary, so that a body rushing through it would be like a man running through the still air—there would be a hurricane "Aether-wind" in the opposite direction. There then in the ether was supposed to be found the great unmoved immovable, to which all motions were to be finally referred.

A very natural question was, "How are we, how is the earth moving in this 'stagnant' ether?" In a letter directed to David Todd by Maxwell, March 19, 1879, shortly before his lamented death (November 5, 1879), and published in *Nature* shortly after (January 29, 1880), a method of astronomical attack upon the problem was indicated, and it was remarked that the time of passage of light forth and back between two points A and B would be longer if the points were in motion than if they were at rest; but inasmuch as the difference would depend on the reciprocal of the squared velocity of light, Maxwell thought it would be "quite too small" for possible observation. In this latter judgment he was at fault. In 1881 Albert Abraham Michelson, a young German, since become the especial boast of American science, performed at the Astrophysical Observatory at Potsdam a modification of Fizeau's famous experiment, by which the motion of the earth in the stagnant ether should have been distinctly revealed in the displacement of certain so-called interference fringes of yellow light. The ingenious experiment consisted in superposing two beams of light that had made equal trips forth and back from two mirrors, one of the trips in the direction of the earth's flight in its orbit, the other at right angles thereto. It was like comparing the time of swimming directly across a stream with that of swimming the same distance directly

against, and then directly back with, the current. To the great surprise of the whole tribe of physicists, the experiment, though executed most successfully and with exceeding care, did not show any motion of the earth relatively to the ether. Some doubt, however, remained, for Michelson had supposed that the time of the light's transverse passage would be "entirely unaffected" by the earth's orbital motion; on the contrary, it would be lengthened by almost exactly half as much as would the time of passage along the earth's orbit. In fact if l be the length from the source, S, to the first mirror, M, v the velocity of the earth, c the velocity of light in vacuo (always and everywhere the same), T the time of passage to the mirror, T_1, the time of reflection back to S, then in the forward flash the light overtakes the mirror, with the effective speed of $c-v$, hence $T=\frac{l}{c-v}$; but in reflection the light meets the mirror, with the effective speed of $c+v$, hence the time $T_1=\frac{l}{c+v}$; the sum of the two is the whole time or $T+T_1=\frac{l}{c-v}+\frac{l}{c+v}=\frac{2\,l\,c}{c^2-v^2}$, and the total distance covered is $\frac{2\,l\,c^2}{c^2-v^2}$ or $\frac{2\,l}{1-\frac{v^2}{c^2}}$. In the transverse passage the distance each way, forth and back, is the hypotenuse of a right triangle, its length is $l/\sqrt{1-\frac{v^2}{c^2}}$ and the total time $2\,T^1$ is not $\frac{2\,l}{c}$ (as if the earth were at rest) but

$$\frac{2\,l}{c}\Big/\sqrt{1-\frac{v^2}{c^2}}=\frac{2\,l}{c}\left\{1+\frac{1}{2}\cdot\frac{v^2}{c^2}+\frac{3}{8}\cdot\frac{v^4}{c^4}+\cdots\cdots\cdots\cdots\right\}$$

$$\text{whereas } T+T_1=\frac{2\,l}{c}\cdot\frac{1}{1-\frac{v^2}{c^2}}=\frac{2\,l}{c}\left\{1+\frac{v^2}{c^2}+\frac{v^4}{c^4}+\cdots\cdots\right\}$$

In this case the earth-speed v is only $\frac{1}{10{,}000}$ of the light-speed C, hence $\frac{v^2}{c^2}=10^{-8}$, and $\frac{v^4}{c^4}=-10^{-16}$.

This latter is far too small for any observation, hence it is neglected, and the difference of the two times is almost exactly $\frac{1}{c}\frac{v^2}{c^2}$ (just half of Michelson's first reckoning). After Lorentz had called attention to this oversight in a penetrating memoir (1886), Michelson along with Ed. W. Morley repeated the experiment in 1887 with far higher degree of accuracy but with the same negative result, and the latter again in 1905 with D. B. Miller with still greater care and precision, and still with the same result. In 1904 Trouton and Noble tested the matter electrically, by swinging up a very delicate parallel-plate condenser, the plates in a vertical plane free to rotate about a vertical axis. If the earth had any velocity relative to ether, a measurable swing should be set up in the condenser by reversing its charge regularly twice in the natural period of swing, just as one may set a pendulum to swinging by pushing it ever so lightly, this way and that, at each instant of its swing through its mid-position. Again it was the unexpected that happened: the condenser would not swing, thereby indicating there was no detectable motion of the earth relatively to the ether.

These and other indications pointing the same way are now accepted as conclusive. So far as we can observe the earth is absolutely at rest in ether; in spite of its spin round its axis, its orbital dance round the sun, its age-long flight among the stars, it does not move relatively to ether.

Of course, thinkers came forward with suggestions to explain this confounding fact. First, it seems, the British Fitzgerald advanced an idea, which met the fate of so many British thoughts, and was past unheeded until presented afterwards independently by Lorentz of Amsterdam—namely, that of the two supposedly equal arms of Michelson's apparatus, the one lying orbitally was shortened by the velocity of the earth's motion; in fact, that every body

shrinks in the line of its motion in the fraction $\sqrt{1-\frac{v^2}{c^2}}$, but not transversely, at all. This would account exactly for the absence of the expected displacement; but such a shortening could not be detected by measuring the two arms, for if laid down side by side they would be found equal, since each would shrink in the same ratio. In fact, no changes in size that affect the measure and the measured in the same way can be detected by comparison. If space and everything in it should suddenly or gradually expand a million-fold, or shrink down to a centillionth, or alternately dilate and contract in any measure whatever, we could not discern it by any conceivable measurement.

Lorentz' hypothesis, which found great favor, was proposed in 1895. In 1904 he expressed it mathematically as a transformation of coördinates, in a memoir on "Electromagnetic phenomena in a system moving with any velocity smaller than that of light." He assumed two sets of axes, one at rest called X, Y, Z; another (X′, Y′, Z′) with the sames axes at first but with the origin moving out along the *X*-axis with the velocity *v* for any time, *t*. Plainly then the new coördinates of any point (x′, y′, z′) would be according to the Galilei-Newtonian Physics x′=x-vt, y′=y, z′=z, and t′=t, since clearly each x would be shortened by the length *vt* while y and z and t would be unchanged. But Lorentz showed that on his new contraction-hypothesis, there would result a very different set of equations, viz:

$$X' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}},\quad Y' = Y,\quad Z' = Z,\quad T' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}},$$

and these equations constitute the famous *Lorentz-Transformation.*

The next year (1905) Einstein arrived at the same transformation, in ignorance of Lorentz's work, and along

quite another line of thought. He did not think of the length along the line of motion as compressed by the motion, somewhat as an accordeon moving in the air, but as a necessary result of the *relativity of motion.* In fact, he laid aside entirely the notion of the stagnant immovable ether, assuming that there was no final system of reference at all, that one system was just as good as any other, if only it moved uniformly on a right line relatively to the other, and he laid at the basis of his whole construction these two postulates which together constitute the so-called Principle of Special (or restricted) Relativity:

> 1. The laws according to which the conditions of physical systems vary are the same when referred to *any* two systems of coördinates that are moving relatively to each other uniformly and on a straight line.
>
> 2. In a coördinate system at rest, light (in vacuo) moves always and everywhere with the same velocity *c,* no matter whether the source of light be in motion or at rest.

These two postulates Einstein then pursued into all of their logical consequences. He found they yielded the Lorentz-transformation and the negative result of Michelson and cleared up other dark points. It resulted that time no less than length and space was relative. This astonishing fact comes to light in many ways, as thus: Suppose there are two points A and B on a straight track, and let the midpoint M between them be exactly determined. Now suppose two flashes, one from A, one from B, are perceived as one by an observer at rest at M; then for him the two flashes are (psychologically) simultaneous. But now suppose at the instant of flash an observer carrying his coördinate system with him, or on a moving train, was gliding through M towards B with half the velocity of light then

he would meet the flash from B 1/3 of the way from M to B, and the flash from A would catch up with him just as he reached B. He would not perceive the flashes as simultaneous, they would be separated by an interval of time; this interval would vary with the speed of the observer. If this observer moved with the speed of light, he would meet the flash from B halfway and he would not be overtaken at all by the flash from A; he would not see it, for him it would not exist. Hence it appears that simultaneity is only relative; that just as a point is describing variously shaped and colored paths in the various coëxistent or coördinate spaces or coördinate systems, so the same event is happening at all sorts of times in these various spaces or coördinate systems. Time has meaning not at all in itself but only in the various spaces or coördinate-systems, and has different meanings in different systems.

This fundamental transformation of Lorentz and Einstein discloses various astonishing relations. The shortening of all lengths in the line of motion has already been mentioned. If we inquire into the times we easily discover that time is similarly affected. Suppose a clock beating seconds in the origin O′ of the moving system X′ Y′ Z′. To an observer in the unmoving origin O of the system x y z the clock will not beat seconds but each interval will be slightly longer, viz, one second divided by the fraction $\sqrt{1-\frac{v^2}{c^2}}$ Again, we are all familiar enough with adding velocities. If you walk forward four miles an hour in a car going 30^m an hour, your total forward speed is reckoned at 34^m an hour; if you walk back it is reckoned at 26^m an hour. But this simple addition does not hold in the new Physics. There the formula is no longer $W = u + v$, but $W = \frac{u+v}{1+\frac{uv}{c^2}}$, where u and v are the velocities added and W is their sum. Now suppose u and v each equal to c, the velocity

of light. Then we get $W = \frac{c+c}{1+\frac{c^2}{c^2}} = \frac{2c}{2} = c$. That is, the sum of two velocities each equal to the velocity of light is still only the velocity of light. If from a gun moving with the speed of light a ball were discharged with the speed of light, as observed from without, it would not leave the gun. No additions to the speed of light can increase that speed. No speed greater than the speed of light is observable. If a body, as this rule, were to start straight up at a speed of 161,000 miles a second, its length would become only half, for the value of the ratio $\sqrt{1-\frac{v^2}{c^2}}$ would be ½. If its speed were to increase up to c, the speed of light, its length would cease to be, it could no longer be observed. Thus it appears that this speed c plays the part of infinity, it cannot be exceeded—which suggests the most heart-searching suspicions.

You are ready to say, "Oh! these things are not so, they merely seem so. The length seems shorter because it is moving so fast, the clock runs slower, because it takes some time to flash back the sign of its finger on the clock-face, the body ceases to be when moving away with the speed of light, because any flash issuing from it is borne away as fast as it approaches us and so can never reach us." This objection seems to be just, but it does not touch the nerve of the matter. For these seemings, these appearances, are after all the only evidences or signs that we have of any physical existence. The doctrine of relativity says we must take these apparent times and spaces at their face values; we can do nothing else, for we have no absolute time or space with which to compare them. As they seem so they are; there is no court of higher jurisdiction to which we can appeal.

So much for the Special-Relativity theory. It is concerned solely with what we might call inertial or Galileian

realms, as the inter-stellar spaces where Galilei's law of inertia holds, and every body is at rest or moving uniformly on a right line. You have observed that not a word has been said thus far of any force, or mass, or attraction, or acceleration, or rotation (except in speaking of possible changes of axes). But all these things are matters of experience and must be taken into account. The result therefore will be the Universal-Relativity theory. Before attacking this a word or two concerning certain profound thoughts of Hermann Minkowski, the most brilliant of recent analysts. They were expressed in a wonderful address at the 80th Meeting of the Society of German Physicists and Physicians at Cologne, September 21, 1908. Einstein has appropriated these thoughts and avows that but for them his own ideas would never have left their cradle. It had long been known that in the general Equations of Transformation there was a so-called Double Invariance: the forms of the equations of motions were not changed by any change of coördinate axes, the equations held just as well for one set of axes as for another, and secondly the forms were not changed by any uniform motion of the axes, but held as well when the axes were moving uniformly as when they were at rest. It was Minkowski's great achievement to bring out the full meaning especially of this latter invariance. He begins with the obvious fact that time and space are known in experience only together, never apart. All that happens, happens somewhere and somewhen. Also, in the equations of motion the time enters like the coördinates x, y. z. Accordingly, any point in space at any time is supposed by Minkowski to be specified not by three coördinates (x, y, z) but by four (x, y, z, t), t, of course, meaning any time-value. It is not merely a point *there* but a point *there-then,* and is called a *World-point.* The whole life, the whole career, of such a point would be the whole assemblage of all its pos-

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sible theres and thens, or better, its *there-thens,* and would be a line not in space simply nor in time simply, but in space-time in a world not of 3 but of 4 dimensions. Such a line is a *World-line.* The whole of Space-Time, the whole of world-history, consists of the totality of such world-points and world-lines. But as we could not observe a single point in space or time, any single world-line would also escape us, and our observations must be wholly of the intersections of world-lines, which make up for us the whole of experience, the totality of world-events. Thus is formed a single static arrangement of all the possibilities of the universe around us, a simple four-fold picture of all that is or was or will be. In this 4-dimensional world-canvas this present meeting will appear as a highly complex knot of intersecting world-lines countless in number; any moment will appear as a cross-section of the gigantic knot. This section will not be a point, as when you cut a line, nor a line, as when you cut a surface, nor a surface, as when you cut a solid, but it will be a solid itself, with the two four-fold extents of space-time fitting upon it and separated by it. So the Conference at Paris appears as another much immenser and worse-tangled knot of world-lines in another world-region; so also the signing of the Declaration of Independence, the crossing of the Rubicon, and so on. The world-line of a point forever at rest would be a line parallel to the time-axis; of a point moving uniformly in a right line it would be a right line sloped to the time-axis. Now, however, since motion is entirely relative we may choose any point as the origin of our system of axes, and in this system at rest the point will be at rest. We ought then to be able to choose our time-axis so as to make any right-line we please parallel to it, and such is the case. In fact, by mathematical ingenuities Minkowski is able to preserve the fundamental invariances of the transformation of coördinates while placing time upon exactly the same footing as any other dimen-

sion, as length or breadth or depth. Accordingly he may then speak of before-and-after exactly as of right-and-left, up-and-down, back-and-forth, without ever making any distinction between time and space, which have thus become melted indistinguishably together. This may confound us at first, but only think that all these distinctions are purely relative. Now the ceiling is up, but on changing our origin to a higher floor the same ceiling would be down. Also by changing the origin and at the same time turning the axes the up-and-downs, right-and-lefts, back-and-forths, would all become completely fused, mixed up quite inextricably with each other. The eggs would be scrambled past all unscrambling, but the new axes would be just as good as the old; the scrambled eggs would combine into new eggs quite as pretty as the old ones. However, these new eggs would not be any better than the old. Precisely so, in Minkowski's four-fold time-space, there remains no longer any distinction of time-axis and space-axes; hence it becomes inept to speak any longer of x, y, z, and t- axes, since, on transformation t may no longer refer to time, which has become blended completely with space. Instead of X Y Z T we may better write X_1 X_2 X_3 X_4 which emphasizes the fact that the axes are quite alike, differing only in the order in which they are named. Now by change of origin and by rotation of axes any one set of axes may be transformed into any other set while still maintaining the same form for the equations of motion, still upholding the invariances that characterize the group of transformations.

Now what does this mean in ordinary speech? It means this: A man on earth and a man in the moon or on the dog-star Sirius would have entirely different sets of axes; up and down would not mean the same for the two, even as it does not mean the same in London and in Tokio, in New York and in New Orleans. Neither would sooner-and-later mean the same thing to the two men; and each

might displace his axes and whirl them about as he pleased. Nevertheless, by virtue of the invariants of the group of transformations, the equations of motion describing the time-space world of each would be equivalent. The 4-dimensional geometries of the two would be the same, and the measurements that they made upon a common object would agree at least as required by the fundamental Lorentz-equations. What then has Minkowski accomplished? He has turned three-fold kinematics into four-fold statics, and mechanics into 4-dimensional geometry. The changing universe of time becomes a single instantaneous universal presentation in 4-dimensions, where past, present, and future appear all alike, all equally real, but none absolute. It is a kind of ecstatic revelation, a stereoscopic vision of eternity in 4-dimensions. We all know how the stereoscope constructs the plane surface of the picture into a three-fold space; somewhat similarly the thought of Minkowski projects all three-fold space along a fourth axis into this hyperstereoscopic vision of unending space and time.

Einstein returned upon his problem of relativity with a mind enriched by Minkowski's ideas, which Minkowski himself alas! did not live to elaborate. Thus far the framework or system of reference was supposed to be a so-called Galilean body, that is, a body rigid and at rest or moving uniformly in a right-line; and all such systems were equally good; the equations of motion were alike for all. Such was the principle of special relativity which declared that the physical description of nature-processes remains the same no matter which of any two systems, S and S′, be employed provided only it be rigid, and the relative motion of the two be uniform and rectilinear. There is *no* preferred system, such as the ether was long supposed to be, referred to which the motions would be true and real, while referred to any other they would be only apparent and unreal. On the contrary, all motions referred to all such Galileian sys-

tems are equally real and equally justified. Now when this postulate is combined with the second, namely, that light moves *in vacuo* always and everywhere with the same speed, then the old Galileian formulae of transformation turn into the new Lorentzian formulate, with all the remarkable results already noted, and many others.

But such passages from one Galileian or inertial body in uniform rectilinear motion to another such, are not the only ones imaginable, just as Galileian bodies are not the only ones imaginable. In fact, such bodies are as scarce as hens' teeth; they are never present in our experience but are only conceptual fictions (à la Vaihinger), though none the less important and even indispensable. But ordinary bodies vary endlessly in their motions. The question now arises: Is it possible to conceive and state the motions of nature so that the description may be equally correct no matter how the system of reference be moving? Such was Einstein's audacious thought, certainly one of the most daring in the history of the human mind. We may say his special doctrine of relativity gave a formal description of nature-motions equally correct whether viewed from the top of the Liberty Statue or from an airplane sailing straight across New York harbor, but now he sought an expression that should still be equally correct if the observer were looping the loop in his airplane or tossed violently in a storm-cloud.

The first impulse to this transcendent doctrine seems to have been given in a difficulty connected with the apparent increase of electro-magnetic inertia of a body in motion, without accompanying increase of weight, of its gravitational mass. These two have never been considered the same. At a certain point between earth and moon, comparatively near the latter, a cannonball's weight would be zero, since the pulls of earth and moon would balance, but the explosion necessary to shoot it off with a given speed

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would be the same as here on earth, for its inertial mass would be unchanged. However, observation ran to the effect that the ratio of the two masses for any body was unchanging and for all bodies the same. A wonderfully delicate experiment of Eotvös in 1891 seemed to show that the ratio could not vary so much as one twenty-millionth. It now occurred to Einstein that inertia and weight are the same thing under different conditions. The like has perhaps occurred to many; I myself have never been able to get rid of the suspicion; but Einstein assumed it boldly as the basis of a daring construction. He saw a man fall from a high building; luckily, the man was not seriously hurt, and he afterwards described to Einstein his sensations during the descent. We all know that actual weight, say at the Equator, is blended of earth-pull and so-called centrifugal force; if the rotation-velocity of the earth were 17 times as great as it is, it would completely annul the earth-pull, and the hugest stone would be lighter than a feather. Now suppose a large chamber in a Galilean field, that is, either at rest or moving uniformly and straight; let an angel or one of Maxwell's "demons" seize the chamber by a hook in the top and bear it aloft with constant pull, that is, with constant acceleration, with steadily increasing speed. How would the man feel? Lighter than a feather before, he is now heavy; he feels in feet and legs the upward push of the floor; he presses on it, and so does everything in the chamber; if he lets anything go, it drops to the floor and all things precisely alike. If he fastens a cord to the top and any body to the cord-end, the body hangs down, the cord is tense. In fact, for all possible observation he finds himself in a gravitation field, solely in virtue of his uniformly accelerated motion.

Of course, it has long been known that the straight path of a projectile (say) in a Galileian field becomes a curve (as a parabola) in a gravitation field; it remained for Einstein

to apply the principle to light. Fifty years ago Maxwell showed that light must press longitudinally, along its rays, against the mirror it strikes; thirty years later this was verified experimentally by the Russian Lebédev and still later, in 1904, and still more accurately by the Americans, Nichols and Hull. But this pressure was due to the light's electromagnetic energy, which endows the light with inertia-mass; did it also have gravitational mass, that is, weight? According to Einstein's so-called *Principle of Equivalence* (of an accelerated Galileian field to a gravitational field) the light must have weight, and a star-ray must be bent sunward in passing near the edge of the sun, so that to an observer on earth it would appear displaced from the sun about 1″.74, just twice the calculated Newton-Maxwell displacement .87″. It is the striking confirmation of this prediction in the British observations on the total eclipse of May 29, 1919, that has so startled the world. But if the light is bent its speed must vary, contrary to Einstein's fundamental assumption; but this was made only for Galilean fields and does not hold for intense gravitation fields. Modifications must be continually introduced into growing scientific structures. For the simpler case, earlier treated, is later perceived to be only a limiting extreme case of a more general and complicated one—into which it is merged not as a contradiction but as a particular under a universal, and it can hope for no kindlier fate.

So much for the uniform gravitation field equivalent to the uniformly accelerated Galileian field. But here on earth the gravity field is not uniform; a body's weight varies from point to point, it is less on the equator, greater towards the poles. So the solution of the problem is only started; how are varying gravitation fields to be treated? Still another difficulty before trying to solve this. We have seen that the actual gravitation field, as at the equator, is a blend of gravitation and rotation. Consider now how this

rotation must affect time-space measures. Suppose a man on a disc that rotates steadily like a merry-go-round. To himself he is at rest but pulled towards the rim; to an outside Galileian observer he is in uniform circular motion. Let the man set down two similar clocks, one on the rim, one at the centre. To the Galileian the moveless central clock runs faster than that on the rim. So also to the disc-man, who would count himself at rest wherever he stood, his clocks would vary in rate from point to point, and time would mean nothing definite, and neither would the phrase "at the same time."

Now let him measure lengths on his disc, and let the Galileian observe him. To this latter the foot-rule at the centre is shortened when laid tangentially on the rim in motion, but not when laid on the diameter of the disc, at right angles to the motion. Hence the rim turns out longer than in a circle at rest, and the calculated value of the perimetric ratio π would be greater than our familiar 3.14159 ..... So then the Euclidian geometry would not apply, the rotation would expand the circumference leaving the diameter unchanged.

If such be the effect of rotation with its so-called centrifugal force exactly the opposite must be that of centripetal acceleration in a gravity-field, as when a mass particle is at the centre of the disc: all radii being in the direction of the varying motion would be shortened in the varying fraction $\sqrt{1-\frac{v^2}{c^2}}$, while all circumference or tangential elements would remain unchanged; hence the unit used in measuring the circumference would be shortened in measuring the diameter, so that the ratio would fall out less than 3.14159....., the circumference would be in effect contracted as regards the diameter, an extremely important fact to be noted hereafter.

The fundamental insistent question now recurs: Can we find a space, a scaffolding, or set of coördinates that shall be related to this world of rotations and central accelerations as the rectangular axial system is related to the Galileian inertial world? In this latter all rectangular systems are alike justified, the description of an event in any one is quite as correct as in any other. In our actual centripetal-centrifugal world are there any corresponding systems, all alike valid for the description of events? That is, in which the equations of motion do not change form on transforming coördinates from system to system? The answer is, yes! They are highly generalized coördinates called Gaussian, from the Göttingen prince of mathematicians, Carl Friedrich Gauss. We all know so-called coördinate paper. Imagine a sheet with the lines as close as you please, each line numbered; each point on the page is fixt as the intersection of two such lines, U and V. Think now of a curved surface covered with such a double system of lines, parallels of latitude and meridians of longitude, similarly numbered and named, each line of each system intersecting each of the other, but none of its own system. Now suppose the paper or the sphere-surface distorted in any way. The double system of lines will be deformed but will determine the points of the surface precisely as before, the point (5, 7) will remain the point (5, 7). The two essential conditions are maintained, namely: Each line of each system intersects each line of the other system in one and only one point, and by making the distance as between two points P and P′ small at will the differences du and dv of the corresponding coördinates may be made small at will.

What has been said of surfaces holds just as well of 3- or 4- or n-dimensional extents; the Gaussians define points in all just the same way. Think of this room as filled with fine spider-threads, 3 systems, u, v, w, from wall to wall; up and down, east and west, north and south—so

that through each point in the room there shall pass one and only one line of each system and in each system let all the lines be numbered exactly. Also suppose the room filled with jelly, to keep each line in place. Now let a September tornado strike the room and twist the jelly mass into any shape but not tear it at any point. Then the lines will be contorted every way, but their intersections remain definite, the point (3, 7, 9) will still be the point (3, 7, 9). Now expand your imagination from the room to all space, with the same systems of lines or Gaussian coördinates, *u*, *v*, *w*. And now suppose a fourth system added, the time-system *t*, giving a four-fold determination in the four-fold time-space world. Then you have the coördinate system used by Einstein. In its use he employs the most powerful and refined mathematical methods of the Italian analysts Ricci and Levi-Civitá, and his stupendous problem is to find such a four-fold system of Gaussian coördinates as shall enable us to represent in equations the complex positions, velocities, and accelerations at any and all instants and on transformations to any other four-fold Gaussian system shall continue to represent them with the same exactness. How he solves the problem by the agency of invariants and covariants and contravariants and fundamental tensors and other such enginery of analysis cannot be set forth now and here; suffice it, that by means of covariance in a certain quadratic differential of four terms, and by means of tensors, that is, expressions such that when a linear relation connects tensors of the same character in one system of coördinates it connects them in all systems, he arrives at laws and equations that do not contradict the Newtonian but include them as special extreme and limiting cases. Thus, Einstein's two equations for the motion of a particle in a plane gravitational field agree with Newton's, except in the addition in one of them of the extremely small term $2\,m\frac{h^2}{r^3}$. But this

happens to be just enough to explain numerically the 43″ per hundred years of rotation of the orbit of Mercury, which Newton's equations left unexplained.

Other examples could be given, but it is not in this higher arithmetical accuracy that the supreme interest lies. The important point is that the jelly-space through which the Gaussians run and always hold their own need not be Euclidian; in fact, the new physics does away with the notion of *force*, along with nearly the whole conceptual apparatus of the elder mechanics and describes the motions of nature as being what they are, and as simulating the operation of forces in Euclidian space, *by virtue of merely taking place in non-Euclidian space*, in space that is curved and crinkled like the surface of the sea—in fact, as it were in a jelly-space that has been twisted and strained but not rent by a tornado. No one balks at a crinkled sea-surface, but many balk at a crinkled sea-solid. The difference is not logical but psychological, in our depictive power. A three-fold space may be curved in a four-fold space quite as well as a two-fold in a three-fold, or a one-fold in a two-fold. So much admitted, and it must be, it then follows that the necessities for gravitative and doubtless other forces in the old physics lay in the hills and valleys of space itself—as Homer would say, "In the numberless folds of Olympus," that is, of the starry expanse.

It is not then that the particle at the sphere—or circle—centre attracts other particles around it, but this particle is a *fiction* (as Vaihinger would say) now to be supplanted by the superior fiction of a curvature in that particular region of space or rather time-space.

This all-important conception of matter as essentially a phenomenon of irregularity in the curvature of space may at first blush revolt the reader, and it may be well to prepare his mind by reference to certain familiar analogous facts. Imagine a circle in its plane; it is smooth, it has the

same curvature at every point; an experience confined within it would have no sense of resistance or urge or irregularity of motion. Similarly on a perfectly smooth sphere-surface, everywhere equally curved. But now suppose the curve not exactly circular but set with minute unequal aufractuosities here and there along its whole length; then in such an experience confined to such a deformed circle a sense of resistance, urge, or unequal motion might well be present. Similarly, if the sphere-surface were not smooth but indented irregularly like the rind of an orange, an experience wholly therein might well involve such ideas as effort, restraint, and varying motion; more accurately, such ideas or experiences might well be interpreted or symbolized under the form of such varying curvature. Now on passing to three-dimensional space the analogy holds perfectly. We must remember that the curvature of such a space is definable and expressible quite as precisely as the curvature of a curve or a surface; and that our three-dimensional space must or may be thought as a border between two four-fold extents, just as the circle is a border between two plane surfaces, its inside and its outside, and the sphere-surface as a border between two solids, inside and outside; also that the plane touches the circle in and out along its whole length, and that solid space touches the sphere-surface, in and out, at every point. Likewise we must think of four-fold space as touching our (three-fold) space at every point and the curvature-irregularities of the latter (its matter-points or regions) as jutting into four-fold space this way and that, forming dents and jags (sharp curvatures) much as the bends, highths, and hollows on the circular curve and the sphere-surface. In such a three-fold space, rugged as a mountain-range, all the phenomena of molar and molecular mechanics, of force, mass, energy, and the like, might well present themselves in experience as characteristic of pure motion.

But the reader will still insist that motion implies *something that moves,* and that this ever persistent something is precisely what he calls *matter,* and is not any flexure of space in which the motion takes place. This objection borrows all its vesture of extreme plausibility from the fundamental assumption of the "natural man" that the reality of the physical world around us consists of *objects* (Kant's "things in themselves") that are what they are independently of mind, soul, or spirit, which merely perceives them and pries into their properties as best it can. Now it is exactly this *Thing-ism* that Einstein's Relativism refutes and expels from Philosophy. Even his doctrine of Special relativity accomplishes this, for it shows that the length (and therewith every dimension of a body) belongs to it not in itself but only as a relation of observer-and-observed and may even vanish under certain conditions of motion. The general theory of relativity takes another long step forward in resolving *thing-character* into *space-property,* in particular, into curvature.

Nevertheless, the reader will remain unsatisfied until he takes the last decisive step towards a positivistic idealism and definitely recognizes that the objective character of the *mover* is only a construct or objectification of psychic activity, of spirit itself. It is not enough to say with Kant that the subject imposes its own conditions or forms of time and space upon the material of sensation freely offered it in experience. This still leaves untouched the independent objective datum, the main source of the woes of philosophy; besides, it is not possible to assign any satisfactory meaning to this imposition of time and space (as forms of intuition) upon the stuff of experience. Nay; the *objects* in question are not the *discoveries* but the *creations* of psychic activity, they are not merely objects but also *objectifications,* for the basic law of spirit-activity is to project, to construct, to image, to *objectify.* Time and space and the categories are

forms or ways in which this continuous externalization takes place, as indeed, in a more refined sense, are all science, all art, and even all history. The varying curvature of space, which gives rise to the notion of permanent objective independent matter, is the image, construct, or objectification of the diversified manifold nature of spirit (or psychic activity) itself, and the conservation of mass and energy is the construct or symbol of abiding unity amid such unending diversity. As Goethe has so impressively declared:

> And all the wide world's wild commotion
> Is endless rest in God the Lord.

To return from a long but important and even necessary digression, we have in fact seen how the centripetal acceleration works in effect a contraction of the circumference and a diminution of the ratio π. A similar change we actually know very well here on earth. From the pole as centre with a quarter-meridian as radius we may draw a great circle on the earth-surface, namely, the Equator. This latter is in length π times the diameter measured *through* the earth, but it is only two times its own arc-diameter measured *on* the earth, through the pole. If we trace a small circle, a parallel of latitude, similarly, we find its ratio to its arc-diameter on the earth is still less than π, but approaches the value π as the circle shrinks up closer and closer round the pole, and very near the pole the value would be indistinguishable (by measurement) from π. You say, "But the arc-diameter is not the real diameter." Very true, if you pass from the earth-surface of two dimensions to the interior earth-space of three. But if you know nothing of that interior three-fold space, and stick to the earth-surface, then your arc-diameter is the only diameter and it functions in your surface-space exactly as a straight line, and on this earth-surface the ratio of circumference to diameter would always be found less than π, but approaching π for

all very small circles. Now as our two-fold earth-surface is curved in three-fold star-space, so may our star-space be curved in four-fold space; and so may our four-fold space-time be curved in a conceivable five-fold space. Accordingly, there is nothing bizarre in supposing that gravitation around and towards the sun is only a dynamical way of stating the pure statical geometric fact that the sun is the name for a certain hump or ridge in space in whose neighborhood the curvature is marked and the circummetric ratio is less than 3.14159. . . . If a path were drawn around his hump l, as the orbit of Mercury, it would be shorter than the corresponding ellipse calculated by Newton and Leverrier in Euclidean space; if we take this ellipse, cut it at one point, straighten it out, and then bend it along on the orbit or path of Mercury round the hump, it will lap over very slightly at the end, and this slight excess of 43″ per century appears in the discrepancy for which the old mechanics could not account, but which the new mechanics explains without remainder.

There are various other considerations that urge upon us this conception of our space as curved and finite. Finite, but not bounded. This important distinction was clearly drawn by Riemann. A circle or any closed curve is finite but has no bounds. A caterpillar may crawl round forever on the rim of a small tub and may very properly fancy he is taking an endless but not an infinite journey. So a football surface is unbounded; an ant might hurry and skurry over it forever without stay or hindrance, and construct an admirable two-dimensional geometry; but the surface is not infinite, it contains just so many square feet, no more. Similarly our space may be boundless but yet finite, having just so many cubic miles.

Grave difficulties meet the supposition that our space is infinite. For if attractive matter were scattered throughout with any finite average density d, however small then

the attraction at any point would be infinite. For describe around any point an enormous sphere of radius R. The attraction at the point would vary as the mass contained,—that is as $d \times R^3$,—divided by R^2; that is, it would vary as R, and would increase with R beyond all limit. This cannot be, hence the mass cannot be spread, however thinly, through infinite space. Hence it has been supposed that the starry universe, the world of matter, is a disc-like island in an infinite ocean of emptiness; and Seeliger especially has tried to modify the Newtonian law of inverse squares; but none of this satisfies. It seems better to assume the indicated spheroidal form of our space, to suppose it is Riemannian rather than Euclidean. Let us suppose that spider lines or light-rays issue from a point P in every direction. On each one measure off a distance R. The ends of all these measures will lie on a sphere of radius R about the center P. Let R increase. In Euclidean space the surface would grow larger and larger, always equally $4\pi R^2$. In Riemannian it would not, but would attain a certain maximum and then shrink down to a point. The surface of a sphere would be $4\pi R^2 \overline{\sin X^2}$; and as this angle X increases from 0 up to 90°, the sin X rises from 0 to 1; as it increases from 90° to 180° the sin X shrinks from 1 to 0. Precisely so, if you describe circles about a pole on the earth-surface, the circle—length will be always $2\pi R \sin X$ (where X is the co-latitude); this sin X increases from 0 to 1 as X increases from 0 to 90°; there the circle is the Equator $2\pi R$; from this on, the circle shrinks as X increases from 90° to 180°, and finally shrinks to a point in the opposite pole. Exactly so, if our space be curved and Riemannian; its volume will be $2\pi^2 R^3$, and the value of R turns out to be simply related to the average density d of matter, namely: $R^2 = \frac{1.08 \times 10^{25}}{d}$ metres. Naturally this squared radius varies inversely as the density of matter,

for *presence of matter* is only a fictitious way of naming the *fact of curvature.*

There is then nothing outré or outlandish about this notion of curved finite space of 3-dimensions and of space-time of 4-dimensions; it is only a natural extension of the notion of a sphere-surface, of a curved finite space of 2-dimensions. But we need no more suppose that this finite space is exactly spherical and smooth-curved everywhere alike than that the earth is exactly spherical and perfectly smooth. On the contrary, the unequal and indeed irregular distribution of matter is only an aspect or symbol of the irregularity of curvature, which is itself the symbol or objectification of singularity inherent in the constructive soul. As the density of matter is never very great, so neither is the curvature anywhere very sharp; space spreads out like a mighty Pacific scarcely so much as fretted by a summer breeze. As Homer says, "Out from the current profound of the River unrippled of Ocean."

Even after leaving out the most that was first written down, the time is well-nigh out and little is left for discussing the "philosophic implications," the main theme originally in mind. Einstein himself never mentions or hints at any; he is obsest with measurements, with his rods and clocks and signals and his purely mathematical apparatus; but Max Planck, the father of the quantum, justly likens Einstein's revolutionary thought to that of Copernicus, as "surpassing in boldness everything previously suggested in speculative natural philosophy and even in philosophical theories of knowledge," in which judgement J. J. Thompson seems to concur. For my own part, I can hardly be a fair, even if I were a competent, judge, so completely and perfectly does the Einstein theory fit in with the general world-view that I have long cherished and am gradually shaping into expression.

For assuredly the rigorous and uncompromising doctrine of universal relativity leaves no escape from the conclusion that the subjects of its treatment are themselves the creatures of the intelligence that so boldly, so audaciously manipulates them, in whose hands they are more plastic than wax, being fused and compounded and contorted at will, to depict and reflect the purely analytic properties of a group of coördinate transformations. If the elements of the time-space world were themselves the inflexible constituents of a world self-existent and independent of the intelligence that was merely contemplating their play on the stage of the physical universe, then indeed all this metrical-mathematical prestidigitation of Einstein's would seem to be as insolent as it was meaningless and absurd. He would indeed seem to be more ridiculous than a dog baying at the moon. But the case is altogether another if the time-space world be itself the obedient creature of mind, the construct of intelligence itself, which has brought it into being as its own self-expression and which is pursuing its age-long, its eternal, task of reducing that construct to an order and system that shall be perfectly rational and self-consistent, that is, that shall symbolize in all its structural details the harmony, the diapason, of the universal soul. Accordingly, the Einstein theory would appear to be the latest and the longest step ever taken in the rational organization and interpretation of human experience. This experience, insofar as it is conscious, seems to begin with the most formless and undifferentiated sensations; it advances to ideas, to *Vorstellungen,* in the creation around us of the perceptual, particularly, of the visual world, and through this it moves on still higher and further to the fashioning of concepts, to the splendid architectonic of science and philosophy. As intuition to the world of percepts, so is theory to the world of concepts. It is in this third heaven of spiritual energy that Einstein and the lamented Minkowski have

celebrated their marvelous triumph. But it must not be supposed that this triumph is by any means complete or final. There are not only many problems still unsolved but many that can hardly as yet be stated. However, the indication seems clear that the physical world must henceforth be regarded as a sacrament, as the outward sign of an inward grace, as the shining symbol of one all-pervasive spiritual activity. Not only then do all the physical sciences in turn merge into mathematics, but mathematics itself, their queen, is merged in logic, and logic is the return of psychology upon itself, and philosophy is the deep-eddying ocean-stream engirdling and engulfing them all.

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LEIBNIZ ON TRUTH AND BEING.

WE take Leibniz on Being first. Being is central in the system of Leibniz. Being, with its various kinds or categories was central in the systems of the great thinkers, from Plato and Aristotle onwards, as was pointed out by Trendelenburg when dealing with the history of the categories in the first volume of his *Historical Contributions.*

Formal logic and mathematics were highly significant for the thought-development of Leibniz, and for the method of upbuilding his system, but this, which relates to the form of his philosophy, is of quite subordinate importance in relation to the contentual side of his system. I am not, therefore, concerned with the rather extravagant claims which have been made, from the formal or methodic side, by thinkers so distinguished as Couturat and others, for logic as heart and soul of his system. The thinkers who have put forward these claims, it is enough now to remark, have themselves, in respects important for my purpose, done a good deal to upset them by admissions of the unfruitful character of the Leibnizian logic. It is not surprising in one who was the greatest mathematician of the rationalist philosophers of the seventeenth century, that his logical theory should have remained mathematical to the core, however concrete the uses to which he put it. Leibniz claimed, strange as it sounds, that his metaphysic was

wholly mathematical, and that mathematicians had as much need to be philosophers, as philosophers had to be mathematicians. And the contention has since been made that the interpretation of his system must depend entirely upon his logic.

For all that, the real or contentual side of his philosophy can still be studied apart from such complete dependence on the place logic had in the formal aspects of his teaching. The excessively deductive method of his system is not denied; but Leibniz was admittedly not the man to sacrifice or efface the real; and while an internal, and not merely exterior, connection must be held between his logic and his metaphysics, it must nevertheless not be overlooked that the materials for his system were drawn from the sciences of the real. This is to give Leibniz credit for seeking to do justice to the claims both of the formal and of the real sciences. It must be remembered, too, that Leibniz was metaphysician more than psychologist, and saw things in their relation to the whole. To him pure ideas—such as those of being, unity, identity—did not depend upon the senses. Being was realized for Leibniz in the monad and the monadistic system. The conception at which Leibniz arrived was that true being is ontological; it was for him metaphysical being, in the sense of activity. From his early studies of Aristotle and the scholastic philosophers, he came to what appeared to him as the finally valid result, viz., that being is strictly metaphysical, as force, activity, substance. That we come upon idealistic elements in it so soon as we enter the peculiarly metaphysical territory, need not, it has been urged, affect the epistemologically realist position or relation. Thus matter and extension, space and time, *e.g.*, were, it is contended, only well-founded phenomena, appearing first in the complicated sensuous intuition, but did not arise from the side of the thing itself. There was no question of idealism or realism, while the

particulars of being were in question, but only as being (as such) was regarded, when taken quite universally.[1]

Leibniz is usually described by the elastic term idealist, in the sense that he held the world's evolutionary development to be according to Divine plan or idea, but not as opposed to the realist assumption of the existence of a real, extra-mental world. His is a teleological, not a phenomenal idealism. But it is not altogether without strangeness that Leibniz did so reach the real existence of the outside world, seeing his monads and no "windows" and were so internally determined as to be fundamentally impervious to outer influences. I think we should to-day allow them not merely windows, but doors, and—not least in importance—skylights. What we have to consider is, how all the foregoing positions of Leibniz as to being were reached and supported. Indivisibility, and reality, were, he thought, ideas which must be preserved, and he found them in force. Force was immaterial and indivisible, yet real and active; and it was with this acting force, conceived as a psychic potency, but yet as the only real existent, that he proceeded to construct the Universe. In what Leibniz says of force and matter, and of rest as merely an infinitely small degree of motion, Leibniz may be taken as having paved the way for the energetics of Ostwald. The teaching of Leibniz on "infinitely little" quantities, or the "Infinitesimal Calculus" was also forerunner of important inquiries into the "infinitely little" in other scientific directions. Force was to Leibniz effort, conation "that from which activity follows," and force or effort "exists quite completely at every instant, and must be something genuine and real."[2] Active force is not, in this dynamical system, "bare potency" to Leibniz; it "includes effort" and "passes into operation by itself." For him there is "no body with-

[1] *Cf.* B. Jansen, *Leibniz erkenntnistheoretischer Realist*, p. 25.

[2] R. Latta, *Monadology*, p. 91.

out motion, no substance without force;" activity is for him "of the essence of substance in general," that is, whether material or immaterial. There can, I think, be no doubt that his doctrine of acting force as inherent in all substances, whether corporeal or spiritual, was important and suggestive. Kant expressly approved the theory, but did not seem to know what to do with it. At any rate, he failed to follow it out. But the position of Leibniz that being and working are the same—that every substance works, and that everything that works is substance—put an end, as Höffding rightly remarks, to conceiving the nature of things as quiescent being or unchanging existence.[3]

Leibniz regards ever-during force as "constitutive of substance," and in notion metaphysical. Substance is for Leibniz not one, but many, the universe being, in a manner, multiplied as many times as there are substances. The number of substances, forces, unities, individuals, is infinite. This multiplicity of individual substances is fundamental to his system. His "units of force" were immaterial and indivisible. To him, there was in reality no dead, corporeal being. Its properties render matter active; it has *virtues*. All reality, however, was to him spiritual in kind. For, to him, the intellect or understanding was far above mere sense-perception. Thought that was free of contradiction held for him real significance—an objective validity. The simple, indivisible entities, void of extension —for extension itself is by him reduced to soul—and lacking in parts, were his monads. Each was created. The passivity of each monad constitutes its *materia prima*. But it has also an active principle or entelechy. Each differs from every other, for "there cannot be two things which completely agree in their internal properties," this being forbidden by his Law of Indiscernibles. The monad is solipsistic, unaware of the existence of any other being;

[3] *History of Philosophy*, Vol. I, p. 340.

and yet, though without "windows" it is supposed to cognize by its own perception, the being of other monads. Each is a "living mirror" or a perpetual image of the universe. The monads vitalize all matter, but in varying degrees of intensity. Rational souls are monads, and are marked by activity. Leibniz says he was led to regard the activity principle as a simple, unextended one, and to apply it to the monads, because he had found it impossible to find the principle of a true unity in matter, or in anything merely passive. All created substances were thus, for Leibniz, active, not passive or inert, as they had been for Descartes.[4] Force being for Leibniz unextended and immaterial, substances had to be immaterial, that is to say, purely psychical forces. The essence of his monads or simple substances was just such psychical force or activity. This psychical force he applies to all nature. God is the supreme monad, most vital and active of all. If all the monads were destroyed, Leibniz holds that this central monad would still contain within itself the whole infinity of existence. This shows it is an ideal universe Leibniz is constructing—not the real universe, but one without experiential basis, which he is seeking to impose upon the existing universe. Rational monads are to him God's agents, sub-rational monads are His instruments. The Supreme Substance is for Leibniz "unique, universal and necessary," nothing outside it being independent of it. Also, it is "illimitable" and "must contain as much reality as is possible." In short, in God as the supreme substance, "perfection is absolutely infinite." Thus God is the primary unity and the original substance. Of created or derivative beings, Leibniz holds their perfections to be through God, their imperfections are "from their own nature." But they have a power of aspiration according to the degree of perfection they possess.

[4] Substance was thus *ens praeditum vi agendi*, to Leibniz.

The existence of the supreme and necessary Being is held by Leibniz, as we shall see later, because, if He is possible, He must, in his view, necessarily exist. As He is without limit, negation, or contradiction; His essence is identical with existence. As to the monads, we saw that they are characterized by activity; this activity is the way in which their individuality—a prime concern with Leibniz —manifests itself. Being is, in Leibniz, drawn out of doing.[5] It is, as active, that the monad has clear and distinct ideas; confused and obscure are its ideas, when passive. The field of such obscure apprehensions, with their power of developing into conscious cognitions, is to Leibniz immeasurable, while clear apprehensions constitute but a few points, he thinks, in our whole mental life. These subconscious sensations[6] occupy a curiously central place, due to a real insight, in the psychological theory of Leibniz. To him, the essence of the monad lay in perception and appetition; and the life of the mind consisted in progress from confused to more distinct perceptions.[7] These "small perceptions" have more effect, Leibniz contends, than we think, for, though unconscious or unfelt, they "indicate and constitute," he says in the "New Essays," the identity of the individual, who retains the traces or expressions of his previous states. As applying to his monads, or ultimate principle of being, Leibniz put forward his Principle of Sufficient Reason, his theory of the sameness or identity of Indiscernibles, and the Law of Continuity. But these three are virtually one, the second and third being necessary corollaries from the first, the principle of Sufficient Reason. By that principle the determination of an effect involved not only logical necessity,

[5] *Quod non agit, non existit.*
[6] *Petites perceptions.*
[7] The inherent activity and native endowment of the mind were expressed by Leibniz in his reply. *Nisi intellectus ipse,* to the Lockian aphorism, *Nihil est in intellectu quod non fuerit in sensu.*

but, as it has been put, "the selection of the best out of an infinite number of logically possible alternatives." The principle of sufficient reason was, to Leibniz, the true motive of world-construction. For the sufficient reasons Lewes, it may be remarked, proposed to substitute "ratios or equivalent values."[8] The view of Leibniz, it may be remarked, that there would be no sufficient reason for two indiscernible beings is really extrinsic to the question of their intrinsic possibility. The metaphysical possibility he did not deny. His Law of Continuity rejected a gulf, or *saltus,* between unthinking matter and unextended thought. Monads, he held, could act on each other only ideally, and not by impact; there was no real interaction between them; if they seemed to be governed in any case, it was because the parts acted harmoniously together, as though directed by a central power. The activity or effort of the monads was thus exclusively immanent in its effects. The ideas of the monad followed a law of its own nature. Transitive activity[9] was denied in its very possibility. But this theory of the substantial independence of the monads does not tell how we are to explain the fact that things do, in actual circumstances and conditions, appear directed towards each other, stand in actual dependence upon each other, and are brought, under scientific inquiry, to reveal intimate causal relations. It casts no explanatory light on the causal, law-conformed connection of things. The unity of this pluralism is inadequate. This inter-connectedness of things in themselves is not satisfactorily explained by simply falling back upon a unified World-Ground, Who relates things to each other, and binds them into a world-whole. The truth is, Leibniz took not the trouble to give us a real and ordered definition of what the monads meant to him. Hence Schopenhauer said he could form no clear notion of the

[8] *Problems of Life and Mind,* Vol. II, p. 94.

[9] *Actio transiens.*

monads of Leibniz, "which are at once mathematical points, corporate atoms, and souls." Again he speaks of the monads as "that monstrous identification of two absurdities, namely, of the atoms, and of the indivisible original and essentially knowing individuals, termed souls."

The question of personal agency, including that of causation, was a deeper and more complex affair than Leibniz realized, with his inadequate notion of what his simple substances involved. If all bodies are compounded of the mounds, they might appear to be matter. If they are viewed as parts of substance, they might be taken as metaphysical, since substance is to Leibniz spiritual force. The monad is a living soul or being, and, as such, may be viewed as metaphysical. But again, the monad is the basis of all reality, and, as such, may be regarded as something real. The conception of the monad is so ill-defined as not readily to lend itself to agreement of the possible conflicting views. There is, I think it must be held, a manifest lack of immanent and essential bonds between the monads. The activity of the monads was perceptive and appetitive, representation and appetition being, as we have seen, their fundamental functions; as cognitive or representative, each mirrors all other monads, and each is miniature of the universe. Indeed, one German interpreter of Leibniz carries this so far as to say that the concept of representation is the most important one for the monadistic system of Leibniz. But it seems hardly an adequate fulfilment of the conative emphasis in Leibniz. Perception of the past, or memory, be held to imply real or personal identity. There were, to Leibniz, as many degrees of perception as there are ranks or stages of monadistic development. The lowest stage—that of "naked" monads—was, in his view, one in which cognition is as confused as in the swooning condition or in dreamless sleep. This is the stage of inorganic nature—the mere analogue of spiritual life—

where consciousness is never attained in any but infinitely small degrees, and the form of expression for the monad is that of motion. But this postulation of confused perceptions is not very satisfactory, if they are to be taken as perceptions that are not conscious. Inorganic nature, is, in the view of Leibniz, an aggregate of sleeping, or undeveloped, monads; man is a waked up monad, distinguishable, in his system, in degree rather than in kind.

By his expositions of the law of continuity, and of the harmony existing in the universe, Leibniz rendered no small service in setting out, after his own fashion, the intelligibility of the world and its relative perfection. But his pre-established harmony was really too complete. His theory of pre-established harmony was merely an idea to compass agreement between ideal thought and real being, while he formulated a world of his own, with elements drawn from both. But the results could not be congruous. These, and other considerations, must not be left out of account when we are asked to credit Leibniz with an idealistic metaphysic, in which the world is taken to be composed of psychic values, thought of after the analogy of our own inner experience. Here, of course, it must be recognized that to Leibniz, at the outset, reality could only be comprehended by reason or thought, experience being held of subordinate importance; and it was by conceptual method, and through ontological speculations, that Leibniz deduced his whole monadistic system. And it must be recognized how much freedom from contradiction in a concept, and the category of possibility, counted for him in the passage to real existence. In such ways metaphysics became so much an activity of pure thought, as to be near of kin to mathematics, which latter was all-important to him. To him, also, body and soul were so independent, that "every body acts as if there were no soul, and every soul acts as if there were no body, and yet both act as if each

was influenced by the other." Yet to Leibniz the soul is never found without its corresponding body.[10] And each body is regarded as a multiplicity of individual souls or substances. That each part of the universe should so completely harmonize with every other part, and that without any mutual dependence or interaction, is all a shade too inexplicably perfect and complete. There is more than a touch of the fanciful at times in the philosophy of Leibniz, but his prescience was undoubted in matters that bore significantly on the speculative theories of scientists of much later times. His philosophy is biologic in type; to him there is nothing really inorganic; all nature is living. This, in contrast with the geometric type in Spinoza. But Leibniz spoke, on occasion, of Jehovah as the eternal Geometrician. The prescience of Leibniz predicted, from the law of continuity, the discovery of polyps, which he deducted as a concept. But the evolution of Leibniz was an unfolding of what was potentially present from the first, it was not of the nature of epigenesis. Darwinism, on the other hand, held that "the germs or gemmules of each separate part were not originally performed, but were continually produced at all ages during each generation, with some handed down from preceding generations."

The existence of God was, to Leibniz, ontological, as it had been to Anselm, to many of the Scholastics, and to Descartes. But as against Descartes, Leibniz urged that the concept of the Highest Being must first be shown to be possible, or without contradiction. But Leibniz held that God, if He be possible, exists. The significant thing, for our purpose, is that Leibniz, like those other rationalists, Descartes and Spinoza, believed that he could conclude from concepts to existence. Kant himself appealed to the principle of thought-necessity on the ground that it

[10] Rosmini reproaches Leibniz for inability to conceive spiritual intelligence without body.

is objective, yet Volkelt said that the principle Kant employed against the ontological argument meant absolute scepticism. The ontological proof of Leibniz is a new proof of both aspects of his theory of knowledge, namely, his realism and his rationalism. It throws him back on the Cartesian-like position that one must, by analysis, form a clear and distinct representation of the object. And as to the certainty of other things, Leibniz expressly says he does not think anybody will be so seriously sceptical as to be uncertain of the existence of the things he sees and feels. He also maintains that things are the causes of the representations. In such matters, he occupies very much the ground of the Scholastics, and indeed approximates the positions of many modern realists, like Hartmann, Messer, Külpe, etc. These are, at any rate, the usual or main insistences of Leibniz.

When it comes to the question of the relation of God, the Supreme Monad, to the monads, it cannot be said that His relation is a very direct or active one. Indeed, after all is said, his doctrine of Deity remains rather obscure. No doubt, Leibniz usually declares God the sufficient reason[11] of all the monads, but too often it is really the final cause of a thing which he takes to be its sufficient reason. It does not, in rigid consistency, appear as though his system had any particular need of the God of theism, for by his doctrine of pre-established harmony in the system of the monads, not much seemed left for God to do. Of course, it can be said that the pre-established harmony was itself the work of a transcendent Creator, but the transcendence still appears as of a deistic kind. The effect of this rather arbitrary and artificial pre-established harmony is, I think, softened, and rendered more natural, in the light of some of the recently edited texts of Leibniz, to which I shall refer presently. In these, Leibniz formulates the

[11] *La raison suffisante.*

law of harmony as a universal law and illustrates its actual working in a variety of ways, and its applicability to particular existences. Harmony is the law of being, a law of intelligibility. The notion of harmony is a rich and complex one. God is for him the ultimate cause of all the harmony of things. Mind being the principle, he says, of unity, all matter implies a mind. But the rigorous development of the notion of harmony is what characterizes these recently edited texts. The "pre-established" character or aspect of the Leibnizian harmony is, however, objectionable. It seems to cancel freedom, and settle all things in advance, rather than leave the future issues to be worked out in a harmony of wills. Leibniz held that God had not only foreseen everything, but provided for everything. What truth his pre-established harmony did contain was that, in the original constitution of things, there are things that fit into each other without a cause and effect connection. What his theory did not take account of was, that there are natural agents so constituted that they act on each other in relations that are properly termed causal. This his monadistic system excluded, allowing no real interaction. But it allowed to every monad that within its limits it could attain its own determined degree of happiness and perfection—a eudaemonistic tendency, of course. We may alllow the whole theory to be free of logical contradition, but, as applied to the actual universe, it breaks down from sheer complexity. And science will not brook its disallowance of interaction. The monad we are compelled to regard as in reality, and by its very nature, a self-transcending conception, since its agency is found in what is actually an interacting system of mutually modifying agencies. His universe, with its too extrinsic unity, still appears as a too pre-ordained and mechanical concern, wound up too completely at Creation, and doing its pre-

destined work through monads all too exclusively shut up within their own consciousnesses.

Leibniz held a mechanistic view of life. All happens mechanically in Nature, he says, but the mechanical laws spring out of a higher law of purpose. It was very distinctive of the philosophy of Leibniz that it sought to harmonize the mechanical cast of the universe with the teleological view of it, and this he did by subordinating mechanicism to teleology. In this reconciling aspect, Lotze is just a more modern Leibniz, only showing advance in his theory of soul. But already, by his own definition, Leibniz made the monad a "spiritual automaton," for the mechanical laws applied, in his view, to living beings and not merely to nature-events, and he denied the monad any knowledge of outside existence and order. No cosmos could possibly arise out of such wholly exclusive beings as his monads, if he had rigidly and consistently carried out his own postulations. However, he found the Divine aims to be realized in Nature, and this through the uninterrupted interconnection of all things. For the individual being or monad constituted a little world by itself, yet its development was supposed to be in harmony with the development of all other beings or monads. For they all had a common origin. They proceeded from God by an emanation or fulguration. But it is yet the mechanical interconnection of Nature with which the philosophy of Leibniz is concerned. Leibniz held God to be the first reason or cause of things, absolutely perfect in power, wisdom, and goodness, and declared the Divine purposefulness in Creation. All the Divine working is, in his view, in accordance with the principle of sufficient reason; for if God "has chosen one less good, there would be something which might be improved in His work." Hegel, however, thought that Leibniz had merely asserted, but not proved his position. The whole monadistic system of Leibniz, with its interesting blend of specu-

lative freedom and scientific restraints, is the result of his ingenious attempt to harmonize Plato with Democritus, Aristotle with Descartes, and the Scholastics with the moderns. But it is honeycombed with inconsistencies, as might be expected. It does not satisfactorily explain how his self-acting, immaterial force can really constitute the plurality of real, imperishable entities with which it has to do, so wondrously dowered as they are supposed to be. It does not explain the extended perceptions of those wholly unextended entities, the monads, nor the other unexplained matters already noted. And there is, further, no adequate account of what, after his manner, we should have to call the world's pre-established discord—its deep moral evils, disharmonies, and irrationalities. Leibniz seems to have thought a great deal more could be deduced by conceptual thinking than was at all possible in the case of the real world. Kant saw how impossible this sort of procedure was, without sense-material being properly given in experience. But Leibniz did not see how his psychical substance, as such, could never possibly produce the world. Psychic force, without raw material of some sort to work upon, could do nothing of the kind. And of the causation necessary to effect the changes in phenomena, Leibniz had no satisfactory account to give. Nor is it matter for surprise that exception has been taken to the way in which Leibniz, so finely and purely intellectual in his great mathematical and metaphysical speculations, yielded to a pragmatic tendency in the latter, and wrote his "Theodicy" in a too practical and accommodating spirit. He also, I may add, describes in this work the Divine intellection in a very human and gratuitous fashion, ascribing to the Divine Mind a plurality of acts which are the merest postulations of the philosopher himself. But there are good things also advanced by Leibniz. He suggestively remarks on the Anselmic proof that "if *Being in itself* be impossible, all

beings through another are so also;" and thus nothing could exist. And again he says "if necessary Being does not exist, there is no being possible."

Couturat's estimate of the philosophy of Leibniz is, that it is the most complete and systematic rationalism; that there is in it perfect accord between thought and things, between nature and spirit; that in it reality is wholly penetrable to reason, because penetrated with reason. Panlogism is, in his view, the only term for this metaphysics. That is an admirable estimate of the system of Leibniz from the formal or logical side, but it does not appear to me to take due account of, or at least sufficiently express, the real side of the philosophy of Leibniz—the side of the metaphysics of real being (of absolute and necessary existences and of contingent or real existences), which is so conspicuous a feature of the thought of Leibniz. This is not to deny that the universe of truth is, for Leibniz, wider than the real. He started from what forms the possible to pure thought, that he might determine, in *à priori* fashion, the nature of the real. But that is no reason why we should forget that such concepts as the real, and objective being, and such categories as unity, substance, existence, necessity, contingency, were not to Leibniz mere subjective determinations or thought-forms as they were to Kant. This, in spite of the idealistic and spiritualistic treatment, on the part of Leibnitz, of matter, extension, space, and time.

In the existential world, Leibniz treats of the knowledge of the ego, of God, and of the sense-world, and these require, I have maintained, from the real side, more than the recognition merely of the formal or logical side of the thought of Leibniz. For, taking the whole thought of Leibniz, inclusive of what the most recently edited texts reveal, the logical view and the ontological aspect of things are never entirely separated. Both must be taken, as I have

urged, to get a complete view of his system. This position is strengthened by reference to the texts of Prof. I. Jagodinsky, described and summarized by A. Rivaud.[12] In these, the thought of Leibniz is, in the highest degree, metaphysical or synthetic. The universe is to him a coherent whole, it embraces, in its harmonious character, the totality of the real. To exist is to enter into the harmony of the whole; it is to be harmonious, according to these texts. I may add that a feature of the system of Leibniz, due to its attempts at theodicy, is, that it is marked, in spite of all talk about its pluralism, by a certain monism—the monism of *order*. The unity of the world, as embodying God's purpose, is the strong contention of Leibniz, for his plurality without unity would mean chaos. The pre-established harmony implies one law, which binds all things together in orderly fashion. Leibniz became compelled, in the outworking of his monadistic system, to give up, in reality, the separateness of the monads, and to treat them as depending upon God, and as organized by Him into the unity of a coherent whole or cosmos. Only those monads, it finally turns out, are real, which are not in isolation, but co-exist as component elements in the world, wherein God's purpose is being realized. Lotze carried out more fully this interrelation of things and beings. This, however, Hegel, with his concept of reciprocity, had, in some respects, done before Lotze. But Leibniz, before either of them, had the merit to regard the universe as, in the last result, spirit, and, as such, One and Many. But these, as we have shown, he never reconciled or harmonized in a satisfactory way. It also remains a suggestive inheritance from Leibniz that every monad mirrors, potentially, the universe, and is big with the future. "For a monad," said Amiel, "which is the virtual universe, a whole infinite of time is not too much to develop the infinite within it."

[12] *Revue de Métaphysique et de Morale.* Jan., 1914.

II.

We now come to the views of Leibniz on Truth. In the mind of Leibniz, questions of the validity and of the rise of knowledge were not sufficiently discriminated, a fact still sometimes overlooked by his interpreters. Quite conformably with his time, he marks no real separation between these—that is, between criticism and psychology. On the formal side, or logical expression, of truth, we have the maintenance to which Leibniz attached so much importance;[13] the notion of the predicate is in some way contained in the notion of the subject. This is another way of saying what he otherwise expresses when he insists that the notion of each individual substance includes all its events and characteristics. Also, in the "New Essays" he insists that knowledge is nothing but the perception of the connection and fitness, or the opposition and unfitness, that one finds between two of one's ideas. A very Lockian definition of knowledge indeed. Leibniz also says, in the same work, that it is better to make truths consist in that agreement between the objects of ideas whereby one is included or not included in the other. The logico-grammatical expression of the truth is, for Leibniz, the judgment; in which he is at one with Aristotle, Aquinas, and Kant. The understanding, in his view, relates a representation to an object. Such power of representation belongs as essentially to the monad as does its striving, hence it cannot be without such representations. The consciousness content of a monad, however, grows out of itself, and is grounded in its own being or essence; it is not borrowed from the outside, but is self-created, on the Leibnizian view.

[13] Namely, *praedicatum inest subjecto.*

In this sense, all our knowledge is of an *à priori* character. This still leaves room for our distinguishing between aprioristic elements in this strict sense, as framed by the understanding, and those empirical elements, which arise from sense-perception. The connection of the notion of the subject with that of the predicate is supposed by Leibniz to be known by the understanding from the mere comparison of the ideas or thought-content. This means such an *à priori* judgment as that just spoken of—an eternal and necessary truth. These necessary truths are so called because they bear the characters of universality and necessity. But if the connection of the two has to be sustained on causal grounds—some sufficient ground or reason—we have an accidental or fact-truth, that is, an *à posteriori* judgment. Contingent truths, says Leibniz, are those which, without doing violence to reason, we may conceive to be otherwise. To say "Socrates was a philosopher" is to assert a proposition which is true, but need not have been so. But Leibniz holds there must be a sufficient reason for contingent truths. And the sufficient or final reason must be outside the serial detail of contingencies, however infinite it may be. Truth of fact, or contingent truth, is truth of experience or perception, and is gained by induction rather than through demonstration. Of such contingent truths, the principle is merely fitness or choice of the best. Necessary truths, on the other hand, depend neither upon instances, nor upon the evidence of the senses; they arise out of reason as absolutely necessary and universally valid, and their contrary would mean a logical contradition, and is therefore impossible. Such are reason-truths[14] or eternal truths. They include metaphysical insights, logical and mathematical truths, and moral laws. Human knowledge is perfected out of the two kinds or sets of truths already described, and rationalism and empiricism are both one-sided, so

[14] *Vérités de raison.*

long as either claims all knowledge as purely from the one or from the other. The Leibnizian distinction seeks to mediate between the two, whose differences are held to be grounded in the quality of reality itself. The necessary truths, resting, as in his view they do, upon the Divine understanding, are unconditional and valid for all worlds. Truth, for Leibniz, exists as a necessity, not outside, but within the Divine Mind.

The distinction of truths of reason and truths of fact, as two qualitatively different kinds of knowledge, has its root in the actually existing difference of the deductively and the inductively elaborated sciences. Identical propositions are the first or *à priori* truths of reason. The first *à posteriori* truths, or truths of fact, are found in the immediate apperception of our own existence, and of our own thoughts and experiences. It was a great merit in Leibniz to have so successfully devoted himself to exhibiting the character and vindicating the validity of the necessary and universal truths. For this task, his rationalism, apriorism, and mathematical genius, eminently fitted him. "The human mind," said Leibniz, "is not only capable of *knowing* universal, necessary truths, but of discovering them in itself. There is a disposition, an aptitude, a preformation, which determines our minds *to elicit* these truths, and causes that they *can* be elicited." Hence Sir William Hamilton credited Leibniz with "the first full and explicit announcement" of "one of the most momentous principles in philosophy." These eternal or necessary truths are, to him, truths of reason, which the understanding recognizes independently of experience and whose validity is independent of it.[15] They cannot in his view, be resolved into anything more simple, for you have come, by analysis, to what is primitive. In the case of these necessary truths, it is said from the idealistic side, that their objective validity

[15] *Ex solis principiis mentis insistis.*

arises from the nature of structure of spirit, and that the understanding is here creator of the object, originator of the laws. No doubt, something is to be said for this, from the psychological point of view. But, from the realistic side, it is contended that the human spirit merely recognizes these truths, but nowise creates them, and that they belong not merely to thought-order, but to the metaphysical order of being. Much can be said for this latter contention. That the necessary and eternal truths are laws of being, and not merely laws of thought, that is, are realized, in the nature of things, is seen from mathematical positions, as when we say, for example, that the whole is greater than the part. It must be remembered that the formal logic of Leibniz has its orientation in metaphysics, and that he sought to complete and improve his logic on the basis of mathematics. The transcendent objective validity of the eternal and necessary truths he sets on an ultimate ontological foundation. He holds that, corresponding to them is being, transcendent and independent of thought, in what he calls the inner possibility of things, the essence of things, in other words, the reality of the metaphysical order. It is in pursuance of this line of thought that we find Leibniz saying "the understanding of God is the region of eternal truths" and that "if there is a reality in essences or possibilities, or rather in eternal truths, this reality must needs be found in something existing and actual" to-wit, "in the existence of the necessary Being." "God," said Leibniz, "is the truth in us," for to him the idea of God was that which is most intimately one with the mind.

The contentions of Leibniz for eternal and necessary truths have lost none of their importance for our own time, wherein sundry attempts have been made to convert logic into a psychological discipline, and to reduce truth to the mere satisfaction of a need. But we must hold, with Leibniz, that eternal truths are valid without reference to the

judging individual. Husserl, in particular, has distinguished the logical content from the act of judgment, and insisted that they must not be mixed. He even separates truth and fact; no truth is to him a fact; by which he means, it is not a temporally determined thing. Facts are, he insists, contingent, individual existences; truth is eternal, timeless, independent of the judging individual. It is certain, I think, that if truth were the mere satisfaction of individual need, it would be particular and subjective, variable with our varying needs, contingent not necessary. Truth would be reduced to a thing of mere "individual expediency." "Man," remarked Cousin, "may say, 'my reason,' but give him credit for never having said 'my truth.'". But Cousin lived before the days of the pragmatists, who now "make" truth. This is truth as viewed by the psychological individual, but this will not satisfy the logical individual, who wishes to be in accord and consistency with cosmic reality and continuity—a vastly wider and infinitely more important thing than particular needs. Hence the rationalistic procedure of Leibniz, which is carried on from the knowledge of first principles or the upbuilding of the reason-truths, occupying, as these do, so important a place in his system, independently of experience and sense-perception.[16] All thoughts were, to Leibniz, properly innate; but he improved upon Locke, and held them to be only *impliciter* and potentially in the soul, and produced from itself, not from the outside. Though innate, they had to be learned, or, as he says, drawn out of our own store, and made perceptible. We find them within us in two ways, he says, by light and by instinct. They are innate, since they could not have been generated from sensuous experience, but the theory is not without objectionable aspects if taken wholly without reference to, and co-operation of, external objects. But the innate ideas are,

[16] This is done by means of what he calls the *lumen innatum.*

in Leibniz, often softened into vague terms, such as inclinations, dispositions, habitudes, that really mean very little. But the extra-mental validity of the first truths or principles so derived was held by Leibniz, who therein differs much in his hold from that of Kant. Kant inadequately realized how profound was the thought of Leibniz, although he was so greatly influenced by him, and not least in the matter of the eternal truths.

Clear and distinct perceptions (after the Cartesian principle) are only useful, Leibniz contends, as the concepts we form are run back into their final logical constituents. But, consistently or not, Leibniz holds the logical laws to be the counter-parts of the laws of being. This, in keeping with his theory of pre-established harmony. Truth, is, for him, no copy of reality, but is logically prior to it, as starting from the possible of pure thought, whence it determines the real in *à priori* fashion. Here we come upon the connection between the two parts of this article, for the base and ground of all truth lies in truth of being —that truth whereby a thing is what it is, has not merely the appearance, but the reality of being. Science is occupied with truth of being. Philosophy is concerned with knowledge of being. Truth of knowledge means truth of idea correspondent or harmonious with truth of being.

It is all very well for Hegel to have supposed, in his "Logic," that he had shown the indissoluble unity of experience, and done away with the distinction between necessary and contingent truths, and between *à priori* and *à posteriori*. There is, I maintain, a final unity or harmony of all the aspects and orders of truth, but philosophy must take due account of distinction or difference on its way to ultimate unification. The value and serviceableness of the distinction drawn by Leibniz between necessary and contingent truths are too real to be so easily obliterated by a mere thought unity, which is all the panlogistic scheme of

the Hegelian "Logic" can give us. You are not at liberty to seek a monism in which no real justice is done to the mind's need of definiteness and distinction on the way to a final monism, if a monism be, as I think it is, possible. The monism reached must be a real one, not based on mere concepts, and not the result of repression of this need of the mind. Justice to distinction and difference need not bar the way to a final monism—unity and harmony, but it will secure that the monism is a real oneness with the ground of all being, not a mere thought—unity *in abstracto.*

We have now seen some of the virtues—the breadth and critical insight—of the system of Leibniz, and criticized some of its defects; it was a rationalism that deduced the world too much on *à priori* principles, and raised its constructions in far too great independence of experience. It imposed reason upon reality, instead of grounding reality in reason. Reason, apart from actuality, remains in abstraction. But the real is already rational, far beyond what appears in the system of Leibniz. And only by the rational can the rational be known. But the suggestiveness of that system is not to be denied, and its influence on subsequent philosophy has been far-reaching and fruitful. But a rationalistic philosophy has no need to stand to-day where stood Leibniz; it would be without excuse if it did. But a broad-minded and vital rationalism is still possible, and is for many of us the highest form of philosophy possible. Its fundamental maintenance is just the intelligibility of experience; it founds on experience, but it must be experience in the broadest possible sense, with no cramped, positivist cast about it. But this without neglect of, or injustice to, what the senses can give us. Its faith will still be in reason and its insights, as the highest world-interpreter we have. But it must be living reason, in closest alliance with living will and real feeling, not the rigid, abstract reason of many arid rationalisms of the past, and, as true and vital

rationalism, it must seek the clearest possible conceptions of all that is given to it in experience. Such rationalism refuses to be baffled by the problems of unity in diversity, the One and the Many, change and evolution, life and consciousness, freedom and responsibility, reality and value, purpose and cause, or any of the other vital problems that engage the philosophic mind.

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EVOLUTION AND EPITOMIZATION.

I.

THE most significant characteristic of the sciences at the present time is the extent to which materialistic and mechanistic theories are elaborated and accepted. In saying this one need not ignore the situation in biology, where vitalism still exhibits some vigor; for it will be strange, after all, if the science of biology does not in its own development exhibit one of those mutations of which it speaks, and one of these days, with apparent suddenness, burst forth into a form which will at last give expression to its many latent and recessive mechanistic factors.

It is a problem for philosophy, concerned with the maintenance and development of abiding values, to know how to deal adequately with this trend of the sciences. In the past, it has been possible to depend upon a supposed duality of matter and spirit; but since the conception of spirit, especially in its teleological forms, is beset by difficulties, and since, according to the modern scientific view, no one can think of matter as threatening to prove ultimate, nor as being coarse, nor inert, nor degraded, one is led to suspect that men who never understood either matter or spirit jumped to the conclusion that they were opposites. Sometimes the views of Lotze and his successors are invoked to refute mechanism; but the day has gone by when anything which is admitted to be universal in extent can be dismissed as negligible in importance. Nowadays

mechanism secretes its forces in the wooden horse of psychology, and thereby penetrates the very citadel and sanctuary of values. Of course a concrete value, an actual experience of valuation, is different from an abstract theory about a value; but to fix a gulf between the two, and to declare the latter incapable of interpreting the former, means the appearance of still more troublesome questions about the status of one, or the other, or both. Is it not time that a philosophy which still recognizes the importance of the problem of a sanction for our great values should, instead of trying to undermine materialism and mechanism, or get round them, accept them and try out their constructive possibilities? Perhaps the presence of values and ideas in the midst of a mechanistic world process can be accounted for in a way which will give them added significance by reason of their essential relations to it.

In this paper an attempt will be made to indicate, in a preliminary and sometimes tentative way, some of the steps in the development of such a view. It is obvious that such a view must call for a correlation of the sciences. The correlations usually attempted are either abstract and general, or else fragmentary, or else dependent upon what seems to be an inadequate epistemology. The great example of an abstract and general correlation is the theory of evolution. Fragmentary correlations are found in such combinations as physical chemistry and physiological psychology. A correlation pivoting upon epistemology is attempted when it is said that the various sciences are so many abstractions in which the universe is studied from different points of view.

Of these correlations the most striking is that afforded by the general theory of evolution. Two broad criticisms may be brought against it. From the side of pluralism it may be said that perhaps, even when formulated most scientifically, the theory imparts a false simplicity to the world—the evolutionary process may be only one current in

the vast ocean of the universe. Another criticism comes from the side of philosophy of religion. If it be granted that philosophy of religion has a right to criticize in such matters, is it not significant that this most meaningful portion of philosophy has, after all, gained but little from this most prevalent and important of scientific concepts? From the questions suggested by Huxley's *Evolution and Ethics* to the attempts like that of Royce[1] to give evolution a ground in the Absolute, there seems to run a common motivation, to the effect that the theory of evolution, in one way or another, fails to measure up to the hopes or demands of religion. Only pragmatism and its kindred philosophies seem to hail the theory of evolution with anything resembling joy; and yet pragmatism, with all the data of evolution at its disposal, offers us adventure rather than assurance—forgetting that a process before which the doors of the future stand wide open can destroy its own treasures and devour its own children. There seems to be a need for the development of some line of thinking which, while agreeing with pluralism that we can at most analyze only a portion of the universe, and with pragmatism that absolute assurance is not possible, will, as against pluralism, treat the portion available for study as an organic whole, and, as against pragmatism, make clear, if possible, that there are certain principles which govern the process of evolution and prevent its wandering entirely afield.

The fragmentary correlations, noted above, are too familiar to need comment. The attempts to correlate the sciences by considering them as so many abstractions, so many partial views of the world process, taken from various angles and in accordance with different interests, bear witness to the basic fact that sciences do not develop without scientists, but fail after all to give due recognition to the

[1] *Cf.* J. Royce, *The Religious Aspect of Philosophy* (1885), pp. 238 ff, 460 ff.

overwhelming aspect of the universe, and to what we must suppose are the independent relations of its structures and processes. The vicissitudes of idealism ought to make it clear, if anything can, that the world is not a case of the knowing process, but that the knowing process is one of the interactions between some of the many structures—material, physiological and psychical—of which the world is composed.

Our proposition is, to study these structures and processes in detail, and thereby, while showing the partial nature of all the foregoing correlations of the sciences, to offer a basis for a new correlation. We shall include the knowing process, but regard it as a particular case of a wider group of interactions; and we shall use the data of evolution, but try to show that the process of evolution itself acquires new content and meaning when the data are exhibited as evidences of another process or group of processes which we call epitomizations.

Epitomization has three specific implications which we shall formulate in this paper. It may be defined in general as the occurrence, at the so-called later stages of evolution, of structures and processes which are essentially analogous to those occurring at the so-called earlier stages. Examples may be noted in the resemblances of atoms—if Rutherford's theories are to be accepted—and the solar system; or, of unicellular and multicellular organisms; or, according to some geologists, of an organism and the earth. By formulating the general definition as above, one avoids the question, unimportant for our purposes, whether the so-called earlier and later stages of evolution are correctly arranged, or whether, if a large enough portion of the universe is taken into consideration, all the structures and processes do not develop simultaneously. What is proposed is not so much a critique of the notion of evolution as a new analysis of the evidence which is cited as a basis of

the theory of evolution. The use of the words "essentially analogous," and the method which they imply, can best be defended after some of the data have been examined.

II.

The first of the three implications of the hypothesis of epitomization leads to a kind of realistic monadism. It is the most abstract of the three, although not altogether separable from the other two. It involves more intricate and detailed work than the others, and offers some of the chief difficulties in the way of the development of the general argument. It is not pretended that the investigations here are complete, nor at some points beyond the hypothetical stage; but it is suggested that here is a line of inquiry which, even at the cost of much correction and reformulation, may be expected to remain significant. It is granted that such a monadism can show only one, or at most only a few of the factors which are operative in the world; but it is urged that to disregard this monadism is to disregard the details of the world's structures and processes. It ought to be noted that we are dealing not, as in the case of the older monadisms, with purely speculative entities, but with things which are studied in laboratories and observatories.

The theory of evolution assumes that certain of the constituents of the world as we now know it correspond to, if they do not survive from, other constituents at earlier stages of its development, and that these stages can be arranged in one or more evolutionary series. For example, when it is found that material bodies as we know them are made up of atoms or electrons, it is inferred that independently existing atoms or electrons have, at earlier stages of evolution, antedated those material bodies. Let us call each such quasi-independent constituent of the world a monad, taking the term in a sense neither less nor more

definite than the term "individual"; and, without at present attempting to go far into the metaphysics involved, let us enumerate these monads as a step preliminary to that of indicating a schematic arrangement to which their analogous structures and processes can be reduced.

Beginnings in this case must be determined by the limitations of present-day scientific investigation. Without endeavoring to account for the origin of electrons, we may begin with (1) electrons in an electromagnetic field, where changes of velocity give rise to radiations. After these we may place the more or less familiar series—(2) atoms; (3) molecules; (4) masses of matter, including both the smaller and larger astronomical bodies; (5) our solar system and perhaps other solar systems; (6) larger units, more or less indeterminate, such as globular clusters,[2] our galaxy,[3] possibly some of the spiral nebulate,[4] and perhaps even one or more vast groups of spiral nebulae.[5] Bearing in mind that this is only a preliminary enumeration, and postponing many questions which arise concerning the relations of these monads, let us pass to another series, at least partially dependent upon the former. Here again it is hard to tell where the series begins, and any serial numbering of the monads must be open to question; but we may assume that some of the organic compounds of carbon are sufficiently distinguishable among chemical compounds, and sufficiently important in later developments, for us to consider them as monads. We may call these monads by the general term (..7?) "organic compounds." It is not difficult to find biologists willing to entertain, and even to develop somewhat, the assumption that some infra-cellular

[2] H. Shapley, "Star Clusters and the Structure of the Universe," II, *Scientia*, 26; 355 (1919).

[3] *Ibid.*, III, *Scientia*, 27; 100 (1919).

[4] A. C. D. Crommelin, "Are the Spiral Nebulae External Galaxies?" *Scientia*, 21; 365 ff. (1917).

[5] *Cf.* A. S. Eddington, art. "Nebula," *Encyc. Brit.* (11), Vol. 19, p. 333.

free living forms preceded the cells which we know;[6] but there is great diversity of terminology, and it is not clear what portions of the highly differentiated cells now known to us, if indeed any portions, may be supposed to correspond to the primitive free living forms. If persistence and definiteness of behavior of parts in the highly differentiated cells may be taken as indications of correspondence to certain free living primitive forms, some of these primitive forms may have been of the nature of microsomes[7] or chromatin granules;[8] but these subordinate units of the cells are themselves so complex from the point of view of organic chemistry[9] that it is hardly likely that the primitive forms which correspond to them represent merely the first stage of the synthesis of living matter. It seems better to regard them tentatively as (8?) later products, perhaps enzymic,[10] and quite likely corresponding to the chromatin of highly developed cells. Since in these cells both the genes[11] and the chromosomes[12] in which they are found are fairly well defined, it is quite possible that the term chromatin may include units of more than one degree of complexity.[13] The next two members of the series—(9?) unicellular organisms and (10?) multicellular organisms are evidently monads, within the meaning of the term. To think of the next member means to consider again the organismic theories of society;[14] but it accords with our

[6] E. B. Wilson, *The Cell in Development and Inheritance* (1896), pp. 21-22, and note.

[7] E. B. Wilson, *op. cit.*, p. 237.

[8] E. A. Minchin, "The Evolution of the Cell," in *Report of the British Association for the Advancement of Science*, 1915, pp. 437 ff.

[9] *Cf.* E. B. Wilson, *op. cit.*, p. 240.

[10] L. T. Troland, "The Chemical Origin and Regulation of Life," *Monist*, 24; 92 ff (1914); "Biological Enigmas and the Theory of Enzyme Action, *American Naturalist*, 51; 321 (1917).

[11] See T. H. Morgan, *The Physical Basis of Heredity* (1919), pp. 244-246.

[12] T. H. Morgan, *"The Mechanism of Mendelian Heredity* (1915), pp. 118 ff.

[13] *Cf.* Carl Snyder, "Life Without Oxygen," *Science Progress*, 6 (I); 134 (1911).

[14] For a survey of these theories see F. W. Coker, "Organismic Theories of the State, in *"Columbia University Studies in History, Economics and Public Law*, Vol. 38 (1910).

procedure to suppose that the next member, or more likely the several members which should next be considered, are (11 ?..) social groups of multicellular organisms—plants and animals—and symbiotic populations[15] of various degrees of complexity.

There remain some other entities which are not altogether different from some of the foregoing, and which are partially dependent upon them; these are best known to us in the nervous system. We are again confronted by difficulties in choosing the first members of the new series, but we may take them to be (..12?..) certain cells or groups of cells which become highly sensitive and specialized as effectors or receptors.[16] Along with these, and in combination with them, there appear in the course of development the nerve-cells and nerve-nets[17] which, together with their appropriate receptors and effectors, we shall call (13 ?) "nervous areas"—to consider the nerve-cells apart from their effectors and receptors is to reflect the procedure of the laboratory, not the course of evolution. As the next member of the series we take, for reasons which will be discussed below (14 ?), "simple,"[18] or "isolated"[19] reflex arcs, together with their appropriate receptors and effectors. Many of our nervous reactions are (15 ?) "complex reflexes," with appropriate receptors and effectors. All these enter into, or are differentiated within (16?..) the great complexes which constitute our emotional[20] and instinctive reactions, or—by the help of pur-

[15] A. J. Herbertson, "Discussion on Natural Regions of the World," in *Report of the British Association for the Advancement of Science, 1913*, p. 557.

[16] G. T. Ladd and R. S. Woodworth, *Elements of Physiological Psychology* (1911), pp. 16-17.

[17] *Ibid.*, pp. 17-18.

[18] C. S. Sherrington, *The Integrative Action of the Nervous System* (1906), p. 8.

[19] Ladd and Woodworth, *op. cit.*, p. 146.

[20] A. F. Shand, *The Foundations of Character* (1914), Chapter III.

posive ideas—our sentiments,[21] values, and "selves."[22]

The first implication of the hypothesis of epitomization is that if the series above enumerated be taken to represent even roughly the course of evolution, each monad, which marks a stage of evolution, exhibits structures and processes essentially analogous to those of the other monads. The course of evolution thus exhibits a series of epitomizations, in the first or monadistic sense of the term. That these characteristic structures and processes, which we shall now attempt to indicate, may recall sometimes the illustrations of Spencer's *First Principles* or the *a priori* methods of Hegel's *Naturphilosophie*, or numerous other relics of the past, is admitted, but held to be beside the point, which is whether the monads as defined exhibit the structures and processes or not.

Formulated briefly, the significant structures and processes may be said to be four—(I) *Individuation*, (II) *Interaction*, (III) *Reproduction*, (IV) *Integration*. Formulated in more detail, they include various subdivisions, as below. It may be noted that, although no hard and fast distinction is made, structures are emphasized in (I) and (IV), and processes in (II) and (III). Stated in more detail, the monadic formula is as follows:

> (1) *Individuation* of a monad, in (I-a) a milieu, which may include[23] [(I-a′) monads of stages previously developed, and] (I-a″) other independently arising monads of the given stage. This individuation is at most only relative, because of (II) *Interactions* of each monad with (II-a) the milieu [and (II-a′) monads of stages previously developed] and (II-a″) other monads of the same stage. With the inter-

[21] *Ibid.*, Chapter IV.

[22] W. B. Pillsbury, *The Essentials of Psychology* (1914), p. 352.

[23] Portions of the formula enclosed in brackets indicate structures and processes which appear at later stages of evolution rather than at earlier; if they occur at earlier stages they have not been detected.

actions are associated processes of (II-b) accretion or depletion [and (II-b′) internal, compensating interactions] sometimes involving (II-c) attraction or repulsion, which (II-c′) varies as new interactions ensue; [the result may be (II-d) disintegration of the monad, but sometimes, before this occurs, (III) *Reproduction* of another monad or monads from it] with (III-a) differences between monads which—especially after some of these monads of any stage, say n, have formed monads of stage $(n+1)$, as in (IV-b), below—may be summed up as relative differences between the terms of (III-b) a duality. [Some of these dualities may come to affect the processes of (III-b′) reproduction in a complex way.] Several of these individual monads may form (IV-a) *Aggregates,* within which they are (IV-a′) juxtaposed in space, or (IV-a″) succeed one another in time; or, several of the monads—sometimes those already within aggregates—may form (IV-b) *Integrates.* In both aggregates and integrates each constituent monad modifies all its structures and processes by reason of the presence of the other constituent monads; that is, both aggregates and integrates are marked to some degree by (IV-b′) interrelations of parts and (IV-b″) differentiations of structure. The integrates (IV-b) of any given stage of evolution, n, differ from the aggregates (IV-a) of that stage, in that in the integrates the modifications of the constituent monads are more marked, and in that the integrates (IV-b) of any stage n are the individuated monads (I) of the stage $(n+1)$.

Any monads of a stage n which are not thus integrated may appear, in connection with a stage $(n+1)$, as (I-a′) monads of stages previously developed. The interrelations of parts (IV-b′) and differentiations of

structure (IV-b″) of stage n are correlated, in the stage $(n+1)$, with the interactions (II) and reproductions (III), so that differentiations of structure (IV-b″) of the stage n appear in the stage $(n+1)$ as specializations of function.

[Sometimes new monads of the stage $(n+1)$ seem to arise not so much as the result of integrations (IV-b) of monads of the stage n, above described, but as the result of differentiations (IV-b″) of monads of some other stage, perhaps $(n+2)$. The fact seems to be that both processes are involved—that at some stages of evolution there is no integration of monads without concomitant differentiations of monads of other stages—and that in some cases of the production of new monads, the former process has (IV-c) been somewhat obscured by the latter.]

Before exhibiting the data for this monadistic scheme in such detail as will be possible within the limits of this paper, it may be noted that the scheme, although doubtless abstract enough, may, if taken as a description of the world, supply a metaphysics more concrete than that which deals with space, time, matter, motion, energy, cause, substance, and such concepts. The relations between this metaphysics and the other may be indicated by saying that the notion of substance arises when the milieu and its structures are conceived as uniformly as possible and that the other concepts noted arise when the interactions of various monads in the milieu are considered in one or another of their aspects at the expense of the rest. With a caution against carrying a comparison to detail before a more complete account has been finished, it may be noted that in the above formulation Process II has resemblances to Professor Sheldon's "productive quality,"[24] and Process

[24] W. M. Sheldon, *Strife of Systems and Productive Duality* (1918), Chapters XII and XIII.

IV to Professor Spaulding's "creative synthesis;"[25] and that a way is thus offered in which two principles prominent in contemporary metaphysics may be viewed within a consistent whole.

III.

Coming now to some of the details of the data, let us consider (1) electrons.[26] As was indicated above, they mark the limits of our present knowledge of those earlier stages of evolution through which the universe, or a significant portion of it, may be said to have passed. Nothing is known concerning the evolution of electrons from any previous stage, although there have been many conjectures. In any attempted description of electronic structures or processes the limits of knowledge are very evident. Since the electron is not material, many characteristic resemblances comparatively easy to detect in the cases of other monads must be modified if not given up. Results attained by means of mathematical formulae are often difficult to state in terms of a mechanical model; but there is general agreement that at least any electron may be said to exist in a medium (I-a)—whether the medium is spoken of as aether, or space, or by some other term.[27] The medium is to be regarded as containing a vast number of electrons (I-a″)—there is no indication at present that one electron gives rise to another. That electrons interact (II) with their medium (II-a) is assumed in theories which account for radiations as due to electronic accelerations, and regard radiant energy as motion of a medium.[28] Electrons inter-

[25] E. G. Spaulding, *The New Rationalism* (1918), pp. 448, 500.

[26] In this section the monads as enumerated above, are referred to by the Arabic numerals there tentatively assigned them; their structures and processes are referred to by the Roman numerals, with letters to indicate subdivisions, as found in the monadic formula.

[27] A. S. Eddington, letter in *Nature*, 107, 201 (1921).

[28] *Cf.* H. A. Lorentz, *Theory of Electrons* (1909), p. 52.

act with other electrons (II-a″). They do not grow, as a material body may be said to grow, and their charges, so far as is known, remain constant; but it may be said that with their interactions are associated certain accretions and depletions (II-b), in the sense that their velocities, energies, and sometimes their masses[29] may increase or decrease. The interactions of electrons with other electrons are marked by attractions and repulsions (II-c), which depend upon relative positions (II-c′). Each electron differs from every other, even if the difference be only one of position in space and time (III-a). In the atomic relationships of electrons, there is a duality of positive and negative charges (III-b). The fact that the medium must be regarded as containing a vast number of electrons is another way of saying that electrons may be associated as an aggregate (IV-a). They may also, according to accepted theories, be integrated with positive nuclei—the exact nature of which is at present a matter of debate—into atoms (IV-b), marked by interrelations of parts (IV-b′) and differentiations of structure (IV-b″); the chief differentiation is that into central and peripheral regions. These atoms form the individuated monads (1) of the next stage (2), the structures and processes of which are considered in the next paragraph. It should be noted that, according to the recent work of Rutherford,[30] there is much to be said for the view that the heavier atoms are not built up directly from electrons, but from atoms of hydrogen and helium; the monadic formula has a place for such a process, below.

(2) An atom (1) thus appears in its milieu, or medium (I-a), which may contain free electrons (I-a′) and other independently arising atoms (I-a″). That atoms interact with the medium (II-a) appears from the facts of absorp-

[29] J. J. Thomson, article "Matter," in *Encyc. Brit.* (11), Vol. 17, pp. 892-3.
[30] E. Rutherford and J. Chadwick, letter in *Nature*, 107, 41 (1921).

tion and radiation. That atoms take up and give off electrons is indicated by some of the facts of radioactivity and attendant processes; and that some atoms take up and give off other atoms (II-a″) may be inferred from the above-mentioned work of Rutherford. Such interactions involve appropriate accretions or depletions of the atoms (II-b), and, according to the theory of Bohr,[31] for example, internal and compensating interactions (II-b′)—readjustments of the constituent electrons among themselves. The accompanying chemical attractions and repulsions (II-c) vary as new interactions ensue (II-c′), for example in cases of auxiliary valencies[32] and saturation. These interactions may continue up to the point of disintegration of the atom (II-d), of which the most conspicuous known form is the expulsion of alpha-particles in radioactivity.[33] The atoms come to differ among themselves (III-a); the differences, especially apparent in atoms which make up molecules, may be summed up as differences between the terms of a duality, *i.e.*, electropositive and electronegative, or basic and acidic (III-b). This relation of duality among the atoms, considered in the order of increasing atomic numbers, is expressed by the periodic law. Atoms perhaps form aggregates (IV-a) in some of the nebulae which give line-spectra rather than band-spectra. Atoms form integrates which we call molecules, characterized by interrelations of parts (IV-b′) and differentiations of structure (IV-b″)—the chief differentiation being, again, that into central and peripheral regions. In some of the molecules, according to the octet theory,[34] the differentiation resembles that of an atom; but in the more complex molecules, at least in those

[31] N. Bohr, letter in *Nature*, 107, 104 (1921).

[32] See A. Werner, *New Ideas on Inorganic Chemistry*, transl. Hedley (1911), pp. 35, 68.

[33] See, *e. g.*, E. Rutherford, "Radiations from Exploding Atoms, *Nature*, 95, 494 (1915).

[34] I. Langmuir, "The Arrangement of Electrons in Atoms and Molecules, *Jour. Am. Chem. Soc.*, 41 (I), 934 (1919).

of organic chemistry, the constituent atoms belong more definitely to either central or peripheral regions. Certain inorganic, rather than organic molecules are the individuated monads (I) of the next stage (3), the structures and processes of which are considered in the next paragraph. Molecules often arise not so much as the result of integrations from atoms as of differentiations within such astronomical bodies as we shall consider later; the former process is somewhat obscured by the latter (IV-c), but in either case there are interrelations of parts and differentiations of structure.

(3) In order for a molecule to be an individuated monad (I) it is not necessary that the molecule should arise by a combination of atoms in "free space"; it may arise in an atomic aggregate. The assumption that molecules exist in interstellar or interplanetary space has been thought to account for the absorption of light in such regions.[35] The interactions (II) characteristic of molecular monads resemble those of the atoms. The physicist studies molecules in motion, and finds that they differ in velocities; any case of such difference presents a duality (III-b) of swifter and slower molecules. In conditions like those of a vacuum tube, velocities help to determine the results of collisions between molecules,[36] with the dislodging of atoms or ions which may form the basis of new molecules (III-b′). Molecules may be aggregated into nebulous masses, or—perhaps within these—integrated into other astronomical bodies (IV-b), which exhibit interrelations of parts (IV-b′) and differentiations of structure (IV-b″). These astronomical bodies may be minute or vast—composed of relatively few molecules, like a meteorite, or a huge assemblage, as in the case of a nebula, star, planet, or satellite; these astro-

[35] *Cf.* the theory of "cosmic dust"—S. Arrhenius, *Worlds in the Making*, transl. Borns (1908), p. 108.

[36] *Cf.* J. J. Thomson, *Rays of Positive Electricity* (1913), pp. 52-53.

nomical bodies form the individuated monads (I) of the next stage (4). It is evident that many of these astronomical bodies must be said to arise not so much as the result of integrations of molecules, but as the result of differentiations within still larger units,[37] such as nebulae or solar systems; the former process is somewhat obscured by the latter (IV-c).

(4) Omitting much of the complicated detail concerning the structures and processes of the other astronomical bodies, let us consider briefly those of the earth, which presents a complicated case of the operation of the same general principles. According to the widely accepted planetesimal theory, the earth is one of the results of a series of disturbances produced in its parent body by a passing star.[38] The earth was at first a "planetary knot"[39] (I) in the medium (I-a) which contained free molecules, atoms and electrons, with all the interactions which we have considered (I-a′), as well as other planetesimals (I-a″).[40] The planetary knot interacted with all these (II-a, -a′, -a″), and grew by accretion[41] (II-b). At least in some knots[42] there was also depletion, for the lighter gases were radiated away. The earth's internal and compensating interactions (II-b′) include its rotation, magnetism, rock-flow, circulation of waters, atmospheric currents, stratification, weathering, volcanic eruptions, earthquakes—in short, the structures and processes studied in geology and its allied sciences. All these processes are accompanied by what is usually known as gravitational attraction (II-c), which varies according to the mass and distance (II-c′). It is conceivable that the earth's inter-

[37] T. C. Chamberlin, *The Origin of the Earth* (1916), p. 133.
[38] T. C. Chamberlin, *Ibid.*, pp. 130, 135.
[39] *Ibid.*, p. 132.
[40] *Ibid.*, p. 141.
[41] *Ibid.*, pp. 132, 138.
[42] *Ibid.*, p. 138.

actions as outlined may lead to its disintegration (II-d), perhaps as the result of a collision with some other astronomical body, or of loss of energy and consequent falling into the sun. According to some of the older theories, the earth "threw off" its satellite (III); later theories, as noted above, are inclined to the view that such processes are found only in the parent nebula or sun, where, either with or without the agency of a passing star or other body (III-b′), a process of reproduction of one astronomical body from another or others is thought to have taken place. The individual variations between astronomical bodies tend to be obscured in the case of the earth and other planets, because of the fact that they are more or less coördinated as members of the solar system. Between stars, the individual variations (III-a) may be summed up as those between older and younger, or dwarf and giant,[43] or slower and swifter[44] stars; and we may suppose that the differences between planets[45] are remotely comparable to these. A clear case of integration (IV-b) of astronomical bodies into solar systems would require that central bodies "capture"[46] their satellites, spacing them out in such a way as to result in interrelations of parts (IV-b′), and differentiations of structure (IV-b″)—the chief differentiation, again, being that between central and peripheral portions. It is more in accord with accepted theories to say that solar systems arise not so much as the result of integrations of astronomical bodies, but as the result of differentiations within a vast parent nebula. The fact may be that both processes are in operation, with the former somewhat obscured by the latter (IV-c).

[43] H. N. Russell, "Relations Between the Spectra and Other Characteristics of the Stars, *Nature*, 93, 253, 283 (1914).

[44] *Cf. Ibid.*, p. 286.

[45] See T. C. Chamberlin, *op. cit.*, p. 141.

[46] A. C. D. Crommelin has criticized the capture theory in *Scientia*, 17, 1 ff. (1915).

For (5) solar systems, and (6. . .) larger units such as clusters, galaxies, spiral nebulae, etc., the formula appears to fit so far as researches have been made; but the small amount of available data, and the possibility of varying interpretations make definite statements difficult. It may be noted that the physical and astronomical effects[47] which are now regarded as tending to confirm the Einstein theories may be taken as instances of internal and compensating interactions (II-b) within the solar system, or within one or another of the larger units—perhaps within the finite universe itself as the largest and the all-inclusive monad.

(..7?—11?..) That a formula providing for individuations, interactions with accretions and depletions, disintegrations, reproductions, integrations and differentiations of structure would apply in general to the biological monads is evident, although the manifold details of these applications require more attention than can be given them in this paper. One should note the importance, throughout the biological series, of the milieu (I-a), which now includes not merely the medium which serves as milieu for the stages preceding the biological, but also includes all those stages, and constitutes the 'inorganic environment." Each biological monad interacts with the inorganic environment (II-a), and with the biological monads of stages previously developed (II-a′), and with other monads of the same stage (II-a″); when these are reviewed in detail any given organism is described in its vital relations with the environment, other species, and other individuals of its own species. In the higher organisms the processes involved in internal and compensating interactions (II-b′), *e.g.*, periodicities of food-taking, breathing, rest, etc.; alternations of heightened and depressed vitality; growth; diges-

[47] For a discussion of the evidence see W. H. Pickering, "The Einstein Theories," *Scientific American Monthly*, 3, 292 ff. (1921).

tive, assimilative, circulatory, excretory and muscular systems—in short, the processes studied in physiology—present detailed complications of some difficulty. The terms "attraction" and "repulsion" (II-c) may be applied at least to the chemical processes in positive and negative tropisms;[48] it must be expected that some questions will be raised if these terms are extended to include the physiological conditions of appetite, etc., which accompany the appropriation or rejection of food, although such an extension is in line with mechanistic conceptions. Individual variations (III-a) may be summed up as differences between the terms of various dualities (III-b)—older and younger, active and inactive, well-nourished and poorly-nourished—all associated with differences of katabolism and anabolism which, according to a famous theory,[49] is the chief factor in the differentiation of sex (III-b′) in (10?) multicellular organisms. The aggregation (IV-a) or integration (IV-b) of individuals may give rise to (11?) social groups, marked by interrelations of parts (IV-b′) and differentiations of social structure (IV-b″). Groups contain both plants and animals. Among the animal members, a differentiation almost universal is that between leaders and followers. Sometimes social groups seem to arise not by integration of individuals—which suggests the old contract theory of society—but by differentiations within a larger and more indefinite group. The fact is that both processes are in operation, and that the former is obscured by the latter (IV-c). The inclusion of (11?..) social groups within the biological series means, as was indicated, a certain amount of recognition of the organismic theory of society, which is commonly held to have been discredited.[50] The point here, however, is not whether a

[48] See J. Loeb, *The Dynamics of Living Matter* (1906), p. 138 f.
[49] P. Geddes and J. A. Thomson, *The Evolution of Sex* (1914), Chap. X.
[50] For a criticism of the theory see F. W. Coker, *op. cit.*

social group is an organism, but whether a social group is a monad; and it can hardly be doubted that social groups do exhibit the various processes indicated in the monadic formula. The trouble with the organismic theory of society was not so much in the facts which it cited, even though it was guilty of extravagances; the chief trouble was in the facts that it did not cite. It left little room for the individual personality, and failed to take account of the immensely important psychological factors which everyone must recognize as operative. In short, the organismic theory was an abstraction. Let us not forget that the monadic formula is also an abstraction; all that is here contended is that the monadic formula offers the kind of abstraction with which the organismic theory agrees, and in connection with which the organismic theory can be estimated on the basis of the qualified and partial truth which it contains. Larger societies are often differentiated (IV-b″) into what may be called conservative and progressive groups, although these terms usually imply the intervention of developed values and ideas.

There are several reasons why it is not very apparent that the monadic formula applies in neurology and psychology. The first of these reasons is that the grounds for the distinction between neurology and psychology are often mistaken. The distinction, for us, is not that between the two members of a psycho-physical parallelism, nor between the two factors of an "interaction," but between a series of structures and a series of processes, essentially inseparable, for both of which the monadic formula aims to make provision. That which, from the point of view of integrations (IV-b) and differentiations (IV-b″) of the monads of any stage n, is a structure, is from the point of view of interactions (II), especially internal and compensating interactions (II-b′), of the monads of stage $(n+1)$ a process. In other words, individuation is a matter of

process as well as of structure. "Neurology" is a term which covers a series of structures; "psychology" is a term which covers a series of processes—and the two are essentially inseparable. From this point of view, the distinction sometimes made between structural and functional psychology[51] might well be modified. Since neurology and psychology are for us essentially inseparable, we adopt the term "neuropsychology." The second reason for confusion with regard to the application of the monadic formula is allied with the foregoing; it is our heritage from the long debate about epistemology. According to the monadic formula, the whole epistemological problem may be considered as a special case of interaction of monads of any given stage with monads developed at previous stages (II-a′) or at the same stage (II-a″) of evolution. In general, the process of interaction (II) is cumulative from stage to stage and from series to series. When the monads which interact belong to the series numbered from (1) to (6 ?. .), the interactions are confined, as we saw, to various physical, chemical, or astronomical accretions and depletions. The monads which belong to the series numbered from (. .7 ?) to (11 ?. .) are never out of interaction with the foregoing series—biological organisms are acted upon, for example, by radiant energies, masses of matter, gravitational forces, just as if the organisms were part and parcel of the infra-organic world. But the organisms exhibit, in addition, more complicated structures and processes of their own, for example in the ingestion and assimilation of food and the excretion of waste. The fact that these biological processes are more complicated does not mean, however, that they are out of place in the world, or cannot be accounted for in the regular course of evolution. Once more, the monads which belong to the series num-

[51] *Cf.* G. A. Coe, *The Psychology of Religion* (1916), p. 17; E. S. Ames, *The Psychology of Religious Experience* (1910), pp. 18, 19.

bered from (..12?) to (16?..) are never out of interaction with the monads of either of the two series preceding—nerve substance exhibits physiological, as well as chemical and physical interactions. But, again, nerve substance also exhibits, in addition, more complicated structures and processes of its own, which many writers, abstracting processes from structures, are content to call psychological, or psychical, and make very formidable under such names as consciousness, experience, and knowledge. But, according to the view here put forward, our experience and knowledge of the world is to be regarded as a comparatively recent and very highly complicated case of certain interactions found throughout the whole course of evolution. The fact that our sense-organs, nervous systems and muscles interact characteristically with the environment might be expected from the fact that all monads developed in the course of evolution tend to interact with monads previously developed. The "epistemological problem" is not one which has anything to do with the possibility of knowledge, for the phrase "possibility of knowledge" contains a fundamental contradiction in terms. The problem of epistemology has to do with a series of structures and processes which should be treated as descriptively as those met with in the problem of digestion. "Knowledge is natural." A third reason for confusion with regard to the application of the monadic formula to neuropsychology is that the processes of biological reproduction, especially of embryology, exercise important effects upon the development of nervous systems in all the higher species. For one thing, the course of development of nervous structures must be studied where they can be found, scattered throughout a myriad of individual organisms of different species, and subject to all manner of local conditions. Again, in the case of any one individual, the nervous system develops in the embryo by a process of

differentiation which, even more than in other cases noted, obscures (IV-c) the process of integration. But we hold that these facts do not rob the monadic formula of its significance for neuropsychology; it is recognized throughout our argument that the monadic formula is an abstraction, and does not account for all the relations which we find in the actual, concrete world. Some of these relations, as we shall see later, appear not in conection with the monadic formula but with the other two implications of our principle of epitomization; and with these the facts of neuropsychology acquire a new significance. A fourth reason for confusion seems most formidable of all; it might be called the argument from the egocentric advantage—*i.e.*, that we feel, or experience, or know ourselves to be essentially different from the world about us, the world which we analyze and construct into monadic or other formulas. As a corollary of this we find the view that the fact that we have values, purposes, and ideals sets our personalities apart from the structures and processes of the world around us. This fourth point can best be treated a little later, after we have examined some of the monads of the neuropsychological series in accordance with the formula.

(..12?..) The first members of the new series are hardly distinguishable from those of the series preceding; they are (I) the cells which, lying usually on the outer surface of an animal's body, become highly specialized, or sensitive to specific stimuli.[52] They appear in the midst of a milieu (I-a), which at this stage of evolution has come to include the "medium" of the physical and astronomical monads, the "inorganic environment" of the biological monads, and the *"milieu interne,"* or internal medium[53] of the organism. There may also be other independently arising sensitive cells (I-a″). Sensitive cells interact with this milieu

[52] Ladd and Woodworth, *op. cit.*, p. 16.

[53] Sherrington, *op cit.*, p. 4.

(II-a) and its monads previously developed (II-a′), and also, indirectly, in the absence of nervous conduction, with other sensitive cells (II-a″). That each interaction of these cells involves accretion or depletion (II-b), internal and compensating interactions (II-b′), attractions and repulsions (II-c) and their variations (II-c′) follows from general considerations of cell chemistry, which are not altered by the fact that the interactions of these specialized cells are specific. Sensitive cells may proliferate (III) during the growth of the organism or the regeneration of lost parts. They come, especially after the appearance of nerve cells, to be distinguished as (III-b) receptors and effectors; and perhaps, in the lower animals, the development of the latter keeps rather ahead of that of the former[54] (III-b′). Sensitive cells may be aggregated (IV-a) into sensitive areas which develop at various parts of the organism; or they may be integrated (IV-b) with one or more[55] nerve cells into what may, for the lack of a better term, be called "nervous areas." The process of integration is obscured (IV-c), if not supplanted—especially in the embryos of higher organisms[56]—by a differentiation among the cells already present. In any case, the nervous areas exhibit interrelations of parts (IV-b′)—marked by the appearance of fibrils[57]—and differentiations of structure (IV-b″). The chief differentiation is that into central (conducting) and peripheral (receptor and effector) regions. These nervous areas form the individuated monads (I) of the next stage (13 ?).

(13 ?) According to most authorities the nerve-cell or neuron, rather than its fibrils, is the fundamental unit of the nervous system.[58] It is not quite accurate, however, to

[54] Ladd and Woodworth, *op. cit.*, pp. 16-17.
[55] *Ibid.*, pp. 17 ff. *Cf.* Sherrington, op. cit., p. 309, and note [62] on following page.
[56] *Cf.* Ladd and Woodworth, *op. cit.*, pp. 38, ff.
[57] *Ibid.*, pp. 18, 40-41.
[58] *Ibid.*, pp. 113-114.

consider the nerve-cell by itself as an individuated monad; for, from the coelenterates[59] to the human embryo[60] the nerve cells appear only between or in the midst of other cells with which they function.[61] Accordingly, we take as the individuated monad (I), the "nervous area"—one or more nerve-cells with appropriate receptors and effectors. Such a nervous area, in which there is only one nerve-cell, is sometimes called an "arc," although it is important to note that the use of the term arc varies.[62] Nervous areas containing more than one nerve-cell are found when any part[63] of the nerve-net of a coelenterate, for example, is activated; but individuality is a matter of process as well as of structure, and the term nervous area should not be applied to the whole nerve-net except when this functions as a whole. The essential mark of the nervous area, as distinguished from later individuated monads, is that its conduction is indiscriminate and reversible.[64] Such nervous areas (I) arise in the midst of a milieu (I-a)—*i.e.*, they appear in the internal medium of the organism, which is in the midst of the environment, which, in turn, is in the physical and astronomical medium. Since the nervous areas are groups of cells, including sensitive cells, they may be expected to exhibit interactions (II) which are in general like those of any group of cells; especially important are their interactions with the milieu (II-a). That these interactions are accompanied by depletion, if not accretion (II-b) may be inferred from the fact that nerves under stimulation give off carbon dioxide.[65] The internal and

[59] *Ibid.*, pp. 17-18.
[60] *Ibid.*, pp. 38-41.
[61] *Cf.* Sherrington, *op. cit.*, p. 7.
[62] Sherrington (*ibid.*, p. 309) speaks of a case where "the conductor mediating between receptor and effector is itself *a separate cell* intercalated between a receptive cell and an effector cell" as "the simplest *arc* in the organisms which possss a true nervous system." But (*ibid.*, p. 7) he says "the conductor consists, in the reactions which we have to study, of at least *two* nerve-cells,—one connected with the receptor, the other with the effector." (Italics ours.)
[63] Ladd and Woodworth, *op. cit.*, p. 19, first complete paragraph.
[64] *Ibid.*, pp. 19-20; *cf.* Sherrington, *op. cit.*, p. 14.
[65] S. Tashiro, *A Chemical Sign of Life* (1917).

compensating interactions (II-b′) of these nervous areas constitute the process of nervous conduction, which, as we noted, is at this stage indiscriminate and reversible; it is also found that the action of a nerve-net does not differ essentially from that of one of its constituent parts, experimentally isolated.[66] At least in so far as the interactions of these nervous areas are chemical they may be said to involve attractions and repulsions (II-c); variations (II-c′) are apparent, for example in fatigue. The most general difference between nervous areas would appear to be that between the directions of their action, according as these are (III-b) inward or outward—the prototype of the differentiation into sensory and motor. Nervous areas may possibly form aggregates (IV-a) in organisms possessing "diffuse systems";[67] they may possibly be integrated (IV-b) in organisms which, as in flatworms and mollusks,[68] exhibit the beginnings of centralized nervous activity—but in all these cases one must keep in mind that differentiation (IV-c) is more apparent than integration. At any rate, when one compares annelids, arthropods, and vertebrates, there is evidently an increasing tendency for the nerve-cells not to anastosmose as readily as in the nervous areas; in the later species the fibres often run side by side in bundles called nerves, but each maintains its individuality; and the fibrils, at least in vertebrates, apparently do not connect the cells as in a primitive nerve-net, but end in proximity to neighboring cells—there is contiguity rather than continuity. But, in spite of all this, the annelids, arthropods and vertebrates exhibit a type of nervous conduction which is less and less diffuse and indiscriminate than that of the coelenterates with their nerve-nets, or that of the flatworms and mollusks with their partially modified nests.[69] Judging from the definiteness of the

[66] Ladd and Woodworth, *op. cit.*, pp. 18-19.
[67] Sherrington, *op. cit.*, p. 311.
[68] Ladd and Woodworth, *op. cit.*, p. 20.
[69] Ladd and Woodworth, *op. cit.*, pp. 20-22.

process, one would say that in the more highly centralized systems a new type of individuated unit must appear—a type which structurally is in general distinguishable from the fact that it possesses two or more nervous cells whose fibres often run in bundles and whose fibrils sometimes do not unite but form connections across near boundaries. These are essential points in the structure of a reflex-arc,[70] when the term is used in the sense indicated above. Possibly, then, reflex-arcs are sometimes the products of the integration (IV-b) of nervous areas—the fact that nerve fibres so often run in bundles may indicate that each nerve-cell tends at least to maintain its primitive connections with motor and sensory cells or groups of cells, and that nervous areas are still partially traceable in reflex-arcs, although central nerve cells have now been substituted for either effector- or receptor-cells, so that, instead of running directly from receptor through conductor to effector, the impulses now run from receptor through conductor to a central nerve-cell, then from this or some other central nerve-cell to a conductor, and then to an effector.[71] One might perhaps say that central cells serve as effectors for one type of nervous area, *i.e.*, the sensory, and also as receptors for another type, *i.e.*, the motor; something like this seems traceable in certain deliberate combinations of responses with stimuli. But in the formation of reflex-arcs, processes of differentiation within the whole nervous system obscure (IV-c), if they do not supplant, such processes of integration. In any case, reflex arcs exhibit interrelations of parts (IV-b)—evident, for example, in the synopses[72]—and differentiations of structure (IV-b′). The chief differentiation, again, is that into central, or coördinating, and peripheral, or receptor-, effector- and con-

[70] *Cf.* Sherrington, *op. cit.*, pp. 15, ff.

[71] *Cf.* Ladd and Woodworth, *op. cit.*, p. 21.

[72] Sherrington, *op. cit.*, pp. 16-18.

ducting cells. It is customary in dealing with them to speak in terms of process rather than of structure, and use the term "reflex." Reflexes serve to coördinate the activities of the organism, and also are themselves sometimes coördinated. The first grade of coördination is exhibited by the simple reflex, which we take as the individuated monad (I) of the next stage (14 ?).

(14 ?) The simple reflex is, according to Sherrington, "probably a purely abstract conception;"[73] Ladd and Woodworth suggest that a better term would be "isolated reflex," and that the essential point is not that the movement evoked shall be free from complexity, but that it shall run its course uncomplicated by other reactions occurring at the same time[74]—but they agree with Sherrington that probably no part of the nervous system is capable of reaction without affecting and being affected by various other parts.[75] The monadic formula is quite in accord with this, for it considers the simple reflex as possessing only a relative individuation (I) in the midst of the environment and the organism (I-a), and also whatever other reflexes may be operating within a nervous system in relative independence (I-a″). The interactions (II) of such reflexes do not differ in principle from those indicated in the case of nervous areas, although reflex conduction is usually irreversible and has other peculiarities.[76] Reflexes evidently interact with other reflexes (II-a″) in processes of "reinforcement"[77] and "spread."[78] The process of conduction itself may be regarded as an internal and compensating reaction (II-b′) within the reflex—for, whatever be the intimate nature of the process of conduction, some sort

[73] *Ibid.*, p. 8.
[74] Ladd and Woodworth, *op. cit.*, pp. 146-147.
[75] Sherrington, *op. cit.*, pp. 8, 114-115.
[76] *Ibid.*, p. 14.
[77] *Cf. Ibid.*, p. 175.
[78] Ladd and Woodworth, *op. cit.*, p. 169.

of commotion[79] is evidently relayed from one portion of the structure to another. Interactions may continue up to the point of discharge (II-d) of the reflex; discharge does not mean the disintegration of the substance of the nerve-cells, beyond the possibility of their incorporation in succeeding reflexes, but means that the process of coördination is completed, so that energy from the receptors is at length released into the effectors, and also that the nerve-cells which have figured in the transmission are now available for incorporation into new reflexes, through the breaking up of old synaptic connections and the establishment of new connections. This latter feature is evident in the "principle of the common path."[80] Perhaps one reflex may help to give rise to or determine the course of another (III) through certain residual effects, like the lowering of a threshold[81] in a portion of its arcs. Variations between reflexes may be summed up (III-b) in any of the terms which serve to describe antagonistic reflexes[82] and those involved in processes of inhibition;[83] for one of the fundamental characteristics of nervous conduction is that one reflex is functioning in more or less opposition to other reflexes. Several simple reflexes may constitute aggregates (IV-a) juxtaposed in space, or succeeding one another in time; or, they may constitute integrates (IV-b), characterized by interrelations of parts (IV-b′) and differentiations of structure (IV-b″). It is necessary here to make some distinctions. The general term covering both aggregation and integration of reflexes, as well as interactions (I-a″) of the reinforcing or spreading type, is "combination of reflexes," and the resultant reflexes are sometimes called "compound reflexes."[84] The distinctions

[79] *Ibid.*, p. 281.
[80] Sherrington, *op. cit.*, pp. 115 ff. 144.
[81] *Cf. Ibid.*, p. 184.
[82] *Ibid.*, pp. 135 ff.
[83] *Ibid.*, pp. 83, ff.
[84] *Ibid.*, Lectures V and VI, titles.

are, indeed, only relative; but this is also the case in connection with the other monads. We should say then that when a reflex spreads, but the result remains recognizable as merely a modified form of the original reflex, we have a case of interaction (II-a″, above). When reflexes exhibit simultaneous or successive combination, in which the constituent reflexes remain well-defined, we have a case of aggregation (IV-a). When there is a combination in which none of the constituent reflexes are so well defined, but are rather fused in a larger unified reaction, we have an integration (IV-b). To these integrates we shall apply the term "complex reflexes"[85]—they are the individuated monads (I) of the next stage (15 ?). As in other cases, we must keep in mind that they may develop within the nervous system by differentiation (IV-c) more conspicuously than by integration; but in any case they are marked by interrelations of parts and differentiations of structure. The chief differentiations are, again, into central portions—especially the various tracts and regions of the brain, which become more complex as higher species develop—and peripheral portions—the increasingly complex mechanisms of receptors, effectors, and afferent and efferent nerves.

(15 ?) The complex reflex, in the general outline of its structures and processes resembles the simple reflex—so much, in fact, that we tend to describe our complex reflexes by over-simplifying them. Reflection shows that almost any act of one of the higher organisms, especially of man, involves coördination of a variety of nervous impulses coming from many different receptors and proceeding toward many different effectors. The complex reflex (I) has the same general relation to environment and organism (I-a), to previously developed nervous areas, *e.g.*, a local "diffuse system,"[86] (I-a′), and to other reflexes

[85] Sherrington uses this term on p. 143.
[86] *Ibid.*, p. 312.

(I-a″), as is found in the simple reflexes. That a complex reflex interacts with the organism (II-a) appears from various facts in connection with emotions;[87]; that it interacts with local diffuse systems (II-a′), from the innervation of viscera, heart, and blood-vessels; that it interacts with other reflexes (II-a″), from the differences between perception and sensation.[88] Perhaps some instances of "compensatory reflexes"[89] may be regarded as internal and compensating interactions (II-b′) within complex reflexes. The terms attraction and repulsion (II-c) are not often applied to the operation of these reflexes, although such an application would be justified in connection with electrochemical theories of nervous action. Variations in intensity (II-c′) play a prominent part in the reactions.[90] The discharge and subsequent breaking up (II-d) of a complex reflex does not differ in principle from that of a simple reflex. One reflex may help to give rise to another (III) through "association by similarity"—a small surviving portion may surround itself with new associates.[91] Differences between complex reflexes may be summed up (III-b) in any of the terms used to indicate antagonistic or inhibitory reflexes; a duality practically always found is that between tonic, or postural reflexes and clonic reflexes, or those which introduce a sharper and more definite reaction.[92] This distinction is allied with the distinction between familiar and new elements of experience, important (III-b′) in all cases of association. Several complex reflexes may by aggregation (IV-a)—mere juxtaposition in space or time—form habitual or instinctive complexes;[93] or, by integration (IV-b) form complexes which exhibit

[87] Ladd and Woodworth, *op. cit.*, pp. 523-524.
[88] *Cf.* William James, *Principles of Psychology* (1890), Vol. II, p. 3.
[89] *Cf.* Sherrington, *op. cit.*, p. 203.
[90] *Ibid.*, p. 231.
[91] James, *op. cit.*, Vol. I, p. 578.
[92] *Cf. Sherrington*, op. cit., pp. 231, 65.
[93] *Cf.* Ladd and Woodworth, *op. cit.*, p. 146.

more marked interrelations of parts (IV-b′) and differentiations of structure (IV-b″). To this class of complexes belong the more highly developed instincts and emotions,[94] and, in man, the sentiments,[95] values, and "selves;" these probably represent more than one stage of evolution. Such complexes may develop by differentiation (IV-c) within the organism and nervous system, as well as by integration. The chief differentiations within a complex are perhaps those between interoceptive and proprioceptive components,[96] on the one hand—including the structures of the "sympathetic system"—and, on the other hand, the exteroceptive components, proceeding from the environment, particularly through the distance receptors. In complexes involving the action of distance receptors a prominent differentiation is that into "consummatory reactions" and "precurrent reactions;"[97] this differentiation probably furnishes a prototype of consciously purposive action—the object seen, for example, at a distance, having the function later exercised by a perceived or conceived "end," and the precurrent reactions having the functions later exercised by the various means to an end.[98] According to this view, purpose is a case of organicity; according to the traditional view, organicity is a case of purpose.

(16?) The structures and processes of complexes must be considered briefly. That emotions, sentiments and values accord in general with the monadic formula is fairly evident, although their relationships are very complex. They are each distinguished more or less definitely (I) in the midst of the apperception mass (I-a′, I-a″). They interact with the environment and the organism (II-a), with single reflexes (II-a′), and with other complexes (II-a″)—the last point being evident, for example,

[94] *Cf.* W. McDougall, *Introduction to Social Psychology* (1915), p. 29.
[95] *Ibid.*, p. 125.
[96] *Cf.* Sherrington, *op. cit.*, pp. 129, ff. 317.
[97] *Ibid.*, p. 329.
[98] *Cf. Ibid.*, pp. 326, 330; also E. Rignano, "Attention," *Monist*, 22, 4 ff. (1912).

in the incorporation of emotions into sentiments[99] or of values into new values.[100] The various complexes are changed by the addition or elimination of certain factors (II-b), as well as by changes going on within their own organizations (II-b′). There are marked differences of intensity (II-c). Complexes may be dissolved (II-d), or may give rise by association to new complexes (III), with variations (III-a) which may be summed up in any of the terms (III-b) expressing the fact that complexes contend for possession of the motor centres. Several complexes may exhibit even higher integrations (IV-b) into the great complexes known as selves—or possibly into certain ideal unities worthy of the name of personalities.

The last statement anticipates some very important points in the argument of this paper; for we have as yet treated values only in a very general way, and we have said practically nothing of language or ideas. Until these things which are so intimately involved in our actual experience are more adequately treated, the monadic formula must seem exceedingly abstract. It is our conviction, however, that these matters can be treated in accordance with the monadic formula, and that both our abstract theories and our concrete experience may profit by the treatment. This treatment may be summed up in six propositions, as below. The first three propositions I have treated in some detail elsewhere,[101] and hence shall merely state here.

1. Language, in the form of cries, arises among the individuals of an animal group as a result of conflicts of tendencies within their complexes. 2. Language develops sounds corresponding to the various parts of speech, and employed in articulated sentences, in proportion as the

[99] Shand, *op. cit.*, p. 62.

[100] *Cf.* Coe, *op. cit.*, p. 222.

[101] In Section I of an article entitled "The Implicit Duality of Thinking," which will soon appear in the *Journal of Philosophy*.

actions indicated grow more elaborate, and in proportion as speakers and listeners are more and more separated from the objects and actions indicated. 3. Language reactions are, according to some theories, the essential factor in thinking; but other theories trace thinking to perceptual images, or to conflicts of motor tendencies. In any case, thinking develops by the aid of processes of abstraction, generalization, and the metaphorical use of terms.

4. The fourth proposition is that an idea is essentially a complex reflex. To call an idea a simple reflex would be to espouse one or another of the theories mentioned in 3, above, or each of them separately; but to call an idea a complex reflex does justice to the fact that an idea apparently draws both form and content from language reactions, from perceptions, and from motor tendencies all at the same time. Moreover, an idea, like any complex reflex, follows the monadic formula. In a later paper I shall try to indicate in detail that an idea, in its characteristic formulation as a logical judgment exhibits a relative individuation (I), which in logic is called identity; that a typical case of interactions (II) is afforded by predication; that a process of accretion and depletion (II-b) can be traced in affirmation and negation, respectively; that the general process of reproduction (III) appears at this stage in the process of logical inference, especially in the syllogism with its conclusion formed from the interaction of two reflexes, one of which contains somewhat familiar, and the other of which contains somewhat new elements (III-b′)—in other words, the interaction of universal and particular premises; and that judgments, together with other elements of experience, may be integrated into values (IV-b), or that values may appear as differentiations (IV-c) within a more inclusive and indefinite apperception mass. In other words, one may, I think, profitably go

farther than Bosanquet[102] and some other logicians go in the direction of a "biologizing" of logic. I hope at some future time also to develop the view that the monadic formula can be applied to some at least of the fundamental concepts of mathematics. If all this is true, it indicates that the monadic formula exhibits the structure and processes of thinking. At this point one must beware of subjectivism; for it may be said that the monadic formula is an *a priori* affair, and reflects the thinking process only, and that its use to describe the objective world is an unjustifiable projection. There is, indeed, no answer to this charge beyond that afforded by the general outcome of the idealistic as contrasted with the realistic views; but there may be a way of using the monadic formula not merely to describe the world and the thinking process, but also to conserve some of the finest features of idealism. This brings us to the objection noted above and then postponed —to the effect that we feel ourselves to be different from the world around us. Before considering it let us note briefly the fifth proposition, which will help to lead up to our answer.

5. This proposition is to the effect that language and ideas transform emotional and instinctive complexes into sentiments and values. This transformation is effected in part by language, when the consummatory and precurrent reactions are rendered communicable, and hence more definite, and when social agencies, always active where there is language, make for the recognition of common standards. At the same time thinking renders the consummatory and precurrent reactions more abstract, more general, and more metaphorical, until values tend to be dominated not merely by ideas, but also by ideals.

6. But, in spite of all this description, we undoubtedly

[102] B. Bosanquet, *Logic, or the Morphology of Knowledge* (1888), Vol. I, p. vii.

do "feel ourselves to be different" from the world around us. How can this feeling be reconciled with the monadic formula? The answer is that precisely such a feeling is to be expected, on the basis of the monadic formula. According to the argument of this paper, the course of evolution has, as one of its notable results, the production of human social values. This production has resulted in part from what we have called (IV-b) a series of integrations. In proportion as these integrations have been different from mere aggregations, they have constituted what Professor Spaulding calls creative syntheses.[103] In them, in every case, *old things have united to form new things.* The fact that this proposition often seems impossible is due to two errors; first, the failure to recognize that, if indeed anywhere a whole can properly be said to be equivalent to the sum of its parts, it is only in the abstractions of mathematics, and especially in the number theory. In all other situations, if not in mathematics, a whole *is more than* the sum of its parts. The second error is the failure to see in the notion of newness the implicit relativity without which the notion is simply impossible. There is no such thing as the absolutely new; things that are called new have to be related to former experiences in order to be called anything at all. That old things, under appropriate conditions, unite to form new things is then the very spring of evolution—it is this process which enables the world to get from one stage of evolution to another. At early stages the atoms, molecules, astronomical bodies, etc., constitute successively the new monads. There follows a series in which the cells or the metazoa, for example, are new—they are, we may suppose, made up of monads previously developed, or they contain parts which correspond more or less closely to such monads; and they remain constantly in interaction with all preceding monads.

[103] *Op. cit.*, pp. 448, 500.

Later on, the neuropsychological monads are new, within the meaning of the term; even if the processes of integration are obscured or supplanted (IV-c) by those of differentiation, the processes of differentiation, no less, involve combinations essentially new. At length the cumulative series issues in the development of new complex reflexes and new complexes; and these develop, as we indicated, into sentiments and values. The fact, then, that we feel ourselves different from the world in the midst of which we develop is the very expression of our essential "newness." According to the monadic formula, new monads, new stages, new series, have been appearing throughout the majestic course of evolution. When at length the stage of sustained human values is reached, *we are the new things.*

IV.

Such a monadism, even when reinforced by the important considerations just mentioned, does not completely describe the concrete world. From some points of view it is hardly more than a first approximation to such a description; for it is almost inextricably involved in the detail of structures and processes, and lacks the perspective essential to a more adequate philosophy. But within the monadistic arrangement as outlined there is already implicit what we take to be the second implication of the hypothesis of epitomization. According to this view, there is something more than convenience involved in the fact that the descriptive sciences seem to fall so easily into what may be called cosmogonic, biological, and neuropsychological groups. A consideration of these groups, from the point of view at which we actually find ourselves within them, indicates the following points: 1. The chief gaps in our knowledge occur between the groups as noted, or just outside them. 2. Cosmogonic monads are, in general

earlier to appear than biological, and biological monads are earlier to appear than neuropsychological. 3. Of striking interest is it that at a certain stage, (4), in cosmogonic evolution, as noted (*i.e.*, in one of the planetary members of a solar system—the earth), when o nemight expect the course of cosmogonic evolution as thus far indicated to proceed to further integrations of higher astronomical monads, such as (5), (6), etc., some of the complex molecules within the planet set up a new series of stages, the biological, which, when compared with the cosmogonic series, may be described as a kind of internal deployment. Again, later, at a certain stage (10?) in biological evolution, as noted (*i.e.*, in some of the multicellular animal organisms which are in a group including plants), when one might expect the course of biological evolution as thus far indicated to proceed to further integrations of biological monads, such as (11?), etc., some of the complex cells within the organisms set up a new series of stages, the neuropsychological, which, compared with the biological series, may be described, again, as a kind of internal deployment. This striking similarity is not lost if one thinks of the solar system (5) and the earth (4) as arising through processes of differentiation within a larger aggregate, the primitive nebula (IV-c)—for one may also think of social groups (11?) and multicellular organisms (10?) as arising by processess of differentiation within a larger aggregate of primitive protoplasmic forms (IV-c). Taken altogether, there seems to be a point here which deserves more attention than it has received. 4. Of the nomads which most intimately concern us, the earth is much larger than any multicellular organism or social group; and any multicellular organism is much larger than any complex reflex or sentiment- or value-complex. 5. Once more, of these monads which most intimately concern us, the earth contains the multicellular organisms and social groups—we do

not live on the earth, but in it, beneath its atmosphere; and, again, the multicellular organisms contain the complex reflexes and the sentiment- and value-complexes. 6. Finally, the earth, although more ponderous in its processes, is less delicately adjusted and intricate than the multicellular organisms are in theirs; and the organisms, although more ponderous in their processes, are less delicately adjusted and intricate than the complex reflexes are in theirs.

The monadic formula, exhibiting essential similarities of structures and processes, enables us to observe parallels between any or all of the monads noted. The relations just indicated between the cosmogonic, biological, and neuropsychological series suggest that a parallelism drawn between the structures and processes of the three series as such may be of special significance. Such a parallelism, shown below, leads to the formulation of the second of the three implications of the hypothesis of epitomization, as follows: Essential similarities of structures and processes are repeated, not only in monadic stages, but in the cosmogonic, biological, and neuropsychological series as such, with successive relations of container and contained; on smaller scales; but with higher degrees of complexity—or, as it may be put briefly, *cosmogony is epitomized by biology; biology, in turn, is epitomized by neuropsychology.*

Such a parallelism would exhibit the monads and series thus:

Cosmogonic (Medium)	Biological (Environment)	Neuropsychological (Internal Medium)
(1) Electrons	(..7 ?..) Organic Compounds	(..12 ?..) Specialized Cells
(2) Atoms	(8 ?) Chromatin (?)	(13 ?) Nervous Areas
(3) Molecules	(9 ?) Unicellular Organisms	(14 ?) Simple Reflexes
(4) Astronomical Bodies	(10 ?) Multicellular Organisms	(15 ?) Complex Reflexes
(5) Solar Systems	(11 ?..) Social Groups	(16? ..) Complexes
(6 ?..) Clusters, Galaxies, etc.		

The full development of this principle of epitomization in accordance with the parallelism would require all the details mentioned in Section III of this paper, and a great many more. Without attempting a detailed treatment, we may note briefly some of its significant features. In the first place, the serial numbering tentatively adopted above is carried over. The cosmogonic series is thus made to contain more monads than either of the other series, although the number of successive monads called by the term "social group" or "complex" might justifiably be increased. But there is nothing in the principle of epitomization which would require that the number of stages in each series must be equal; and there are so many questions concerning the individuation of the later monads in all three series that it is better not to attempt further extensions for the present. As new monads are discovered or distinguished in future, it is quite possible that the parallelism may need to be modified; indeed, as it now stands, it represents a line of inquiry rather than an established result—but it is a line of inquiry which offers some striking possibilities in the way of a new correlation of the sciences.

It is noticeable, in the second place, that the parallelism sets over against one another three portions of the world which in the various sciences and groups of sciences are apt to be forgotten or neglected—the "medium" in the midst of which the cosmogonic monads develop and with which they interact; the environment in the midst of which the biological monads develop and with which they interact; and the "internal medium," which sustains similar relationships to the neuropsychological monads. Without attempting to explore the detailed possibilities here, let us note that one may construct at least a rough parallelism between current theories in these three groups of sciences, according as the milieu appropriate for each group is regarded as differing in degree or in kind from the monads

which it contains—as conforming, in a modified way, to the laws which are observable among the monads, or as having laws of its own, to which the monads form departures or exceptions.

The next line of the parallelism as shown above brings us to some monads which have what we may call an intermediary relationship between the milieu and the later monads in each series. Thus electrons, while different from the medium, are still not material bodies; organic compounds, while different from the environment, are not living forms; and specialized cells, while different from the internal medium, still do not have the properties of a nervous system. Yet, in each case, the monad mentioned is utterly indispensable for the structures and processes of the later monads of the series. Apparently, on the other hand, each of these monads in its interactions releases into the milieu something which is more akin to the milieu than to the later monads; the electrons set up radiations in the medium, the organic compounds split off less complex organic, or even inorganic components, into the environment, and the specialized cells probably discharge some physiologically active substance into the neighboring cells.

That atoms, chromatin, and nervous areas are set in parallel affords a suggestion which, taken in connection with what has just been said, may help to lift the reproach which has too long and too unjustly been heaped upon materialism. The atom, as contrasted with the electron, is admittedly a material body; possibly chromatin, and certainly unicellular organisms, as contrasted with organic compounds, are living forms; possibly nervous areas, and certainly some of the later neuropsychological monads are, while they function, conscious. The suggestion, which cannot here be developed in detail, is that matter, instead of being the antithesis of life and consciousness, tends to be their analogue. It is further noticeable that atoms, chro-

matin and nervous areas each exhibit somewhat wide variations in magnitude—the monads of each of these stages are not to be thought of as of uniform size. Atoms consist of one or more electrons and one or more positive nuclei; chromatin consists of organic compounds and nucleic acid in varying proportions;[104] nervous areas consist of one or more specialized cells and one or more nerve-cells. A question which calls for special investigation is whether there is not a significant resemblance between the interactions of atoms, with their positive and negative valencies and their valency electrons; chromatin, in its connection with dominant and recessive characters as conditioned, according to the hormone theory of heredity, by organic products split off from the chromatin of the germ-cells; and nervous areas, with their structures and processes which make possible the reinforcement or the inhibition of one reaction by another. This question involves not only these monads, but also some of the later monads of each series, and is too complex for detailed treatment here.

Concerning molecules, unicellular organisms, and simple reflexes when the last named term is understood in the sense used above, it may be remarked that each of these monads, under conditions known to us at present, is more capable of relatively independent existence than are the monads previously developed, but is less familiar to us than later monads.

The parallelism between astronomical bodies and multicellular organisms gives point to biological and physiological comparisons sometimes found, for example, in the work of Chamberlin, and to the more definite assertions of S. Meunier[105] that there are significant resemblances between geological processes and biological processes. These, again, are too complex to be discussed here; and whether

[104] E. B. Wilson, *op. cit.*, pp. 240-242.

[105] *Les Harmonies de L'Evolution Terrestre* (1908); *La Geologie Biologique* (1914).

the resemblances can be extended to the process of conduction as we have it in the monad of the third series, the complex reflex, must also be left open. The most significant point in connection with the threefold parallelism as indicated is the successive relation of container and contained, the importance of which was emphasized above; the monads of the biological series may be said to arise from some of the molecules within one of the astronomical bodies; and the monads of the neuropsychological series may be said to arise from some of the cells within one of the biological organisms. The parallelism is affected, although hardly vitiated, by the fact that we know for certain one astronomical body which is inhabited by biological organisms, whereas we know myriads of organisms which have nervous systems. There may be some significance in certain facts in connection with these monads which, however familiar they are when taken separately, seem strange when set side by side—that, as astronomical bodies, planets exhaust their energy before the suns on which they depend exhaust theirs; further, that, of biological organisms, the animals, in general, expend energy, while the plants upon which they depend store up energy; and finally that, of complex reflexes, the clonic reflexes run their course quickly, while the tonic reflexes are sustained for longer periods.

We have already noted that the parallelism between solar systems and other astronomical groups; social groups of various kinds; and complexes of various kinds, is not worked out in detail. It is only recently that the individuation of some of these monads has been recognized, and the fact that they transcend our ordinary relationships and ways of thinking renders them very difficult to treat. Possibly, when it comes to be recognized that a social group may function in super-individual ways, there may be recognized also some analogous doctrine to the effect that a

sentiment- or value-complex may function in super-ideational or super-rational ways, which may still be valid. At this point the argument for epitomization might establish contact with intuitionism and mysticism.

A few words may be added concerning the relationships of the three series when each is taken as a whole. In the first place, when the details are worked out, it may be possible to strike totals, so to speak, for each series, and point out an epitomizing relationship between the totals—thus, it can perhaps be said, the cosmogonic universe is epitomized in the biological population of a planet; and the biological population is, in turn, epitomized by the apperception mass of an individual. But for the present this generalization is too remote.

There is a certain significance, however, even in the gaps in our knowledge; for, since the chief gaps in our knowledge occur between the series, or outside them, and since spaces between them must for the present be left open, the above parallelism is at present applicable either in a mechanistic or a vitalistic biology, and in either a behaviorist psychology or a psychology which makes of consciousness a kind of intervening agent. There is a possibility that the parallelism, taken together with the other implications of the principle of epitomization, may reinforce Professor Bergson's suggestion[106] of a line of reconciliation between some of these contending schools. Perhaps the conceptions of mechanism should be employed to describe the chemical origin of the life process, while those of vitalism should be interpreted to signify the fact that the biological series departs from the cosmogonic series in what we have called an internal deployment. In other words, perhaps mechanism is properly a concept of origins, while vitalism is properly a concept of orientations of the biological series with reference to the cosmogonic. Simi-

[106] *Creative Evoluton*, transl. Mitchell (1916), pp. 30-31.

larly, perhaps behaviorism can tell us more of the origins and elementary characteristics of the neuropsychological series, while the older views exhibit more of the orientations and larger relationships of the neuropsychological structures and processes. Of course, the gaps between the series may indicate that here extraneous forces intervene —that some vitalistic impulse produces the structures and processes of the biological series, and some quasi-spiritual force those of the following series. The parallelism would, if necessary, accommodate such extraneous forces; but it is more economical of one's concepts to regard the series as appearing without such interventions. The universe, according to the view here taken, might be said to be like one of the projectiles from a modern long range gun, which, after passing through a certain trajectory, produces a second explosion, and then, after passing through another trajectory, produces a third. In the case of the projectile, one does not need to assume that forces from outside intervene to produce the later explosions; one says, simply, that the projectile is built that way. Nor does one need to assume that there are such intervening forces in the case of the universe; one can say that the universe, also, is built that way—although this illustration must not be pressed to the point of eliciting from it a cosmological or a teleological argument.

V.

The second implication of epitomization, as considered above, does not yet exhibit our concrete world in all the fulness of relations which characterize its parts; the threefold parallelism of the preceding section is still in many respects an abstraction. One further step toward concreteness (although this again is not the final step) may be taken when, instead of considering the cosmogonic, bio-

logical and neuropsychological series singly, in their relations to one another, the first and second are taken together, in their essential connection, and set in parallel to the second and third, also taken together, in their essential connection. This is quite Fechnerian,[107] although the Fechnerian consequences do not necessarily follow from it. When the first and second series are taken together, we have ecology; when the second and third are taken, we have physiological psychology. The third implication of the hypothesis of epitomization is that *ecology is epitomized by physiological psychology;* the living organisms are to the earth as the nervous system is to a multicellular organism. Of course, in the concrete world of actual experience, all three exist in indispensable and very complex relationships; but it is held that this fact adds to, rather than diminishes, the significance of the abstractions indicated.

According to this third implication, taken in connection with the others, we may say that social groups in the history of the earth may be compared to complexes in the experience of an individual. An argument, too long to indicate in detail, may be worked out to the effect that the more definitely organized and established social groups—*i.e.,* the civilized human groups—are comparable to the more definitely articulated and persistent complexes—*i.e.,* the value-complexes; and that individual men within the groups, who are either leaders or followers, are comparable to ideas within the complexes, which are either "ends" or "means." The death of an organism in the earth may be compared to the discharge of a reflex in an organism; a view which recalls the essential point of one elaborated by Royce.[108] We may say that after the death of an organism portions of it are either incorporated into other organisms, or the biological processes are succeeded by

[107] See *Zend-Avesta* (1854), Vol. I, p. 286.

[108] *The World and the Individual,* Second Series (1908), pp. 317 ff.

processes which belong to the cosmogonic series; and, similarly, that after the discharge of a reflex portions of it are either incorporated into other reflexes, or the neuropsychological processes are succeeded by processes which belong to the biological series. The third implication of epitomization has also a bearing upon the organismic theories of society; it might be worth while to see how many of the accredited objections to these theories would be removed if society were to be thought of not as an organism, but as a kind of organ, comparable, as Herbertson[109] has suggested, to a nervous system of the earth.

The general result of the three implications of epitomization is a view that the structures and processes of the universe tend, by a kind of progressive concentration throughout the monads of the three series, to be, so to speak, distilled into the social groups, and, through these, into our values and ideas. This view regards the universe as not merely evolving, with the result that men and their experiences have appeared, but also as epitomizing, with the result that men and their experiences have a significant relation to the whole. At this point the hypothesis of epitomization is in general accord with the views which construe the problem of philosophy of religion as a problem which concerns man's relation to the universe; but the implications of the hypothesis for a philosophy of religion must be left for more detailed treatment. The affiliations of the hypothesis in the history of philosophy are with those fantastic but remarkably widespread and persistent theories of macrocosms and microcosms, particularly with the view that man is a microcosm, or little universe. But the views here put forward do not rest upon this historical basis. The problem of the relation of the universe and man is, for our age, first of all a scientific problem, and calls for the contributions of the great specialists along a thousand

[109] The Higher Units, *Scientia*, 14, 212 (1913).

lines of inquiry. But it is no narrowly specialized problem; it calls also for correlation, for perspective, sometimes for speculation, and for all the resources of philosophy. The aim of this paper has been to raise the question, and to indicate some of the possible steps in its solution. It ought to be possible to decide on scientific grounds whether the construction which is here put upon the data of evolution is justified, and, upon philosophical grounds, whether the result is, or may sometime be worth while.

In connection with this argument, something should be said about the use of the method of analogy. It is our conviction that much that is said against the method is conventional, and is directed against its more obvious abuses; and that, as a method, it has yet to receive its adequate evaluation. Of course it must be granted that there are many differences between the various monads and the various series; indeed, from some points of view, the differences are more important than the resemblances. But still the resemblances are there; and when the monadic formula is developed in detail, many of the differences appear as differences of complexity in later as compared with earlier stages, rather than as differences of essential processes involved. In our argument, reliance is placed not merely upon a number of analogies, although the number is quite formidable and, if one insisted upon a logical justification of the method, might serve as data for induction to a general law of analogy under which particular cases might be subsumed. Reliance is placed rather upon what we have called the essential character of the analogies. It is held that false analogies, common in reasoning, are usually hasty or superficial analogies, but that the analogies indicated in the monadic formula and in the parallelism are essential. They are essential in the sense that, so far as we can detect, the world known to us would not have developed without such structures and processes; or, to put the

same thing in another way, they are essential because they constitute a system of metaphysics. And a consideration of the importance of these structures and processes ought to make for the view that the many striking analogies which the universe admittedly contains should be capitalized, and that new analogies should be looked for, and that old and new together should be developed into an instrument of real usefulness in science and philosophy.

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ART AND THE UNCONSCIOUS.

IT has been said that "Art as the formulation of a unique and distinct psychological attitude is unquestionably one." I am in profound agreement with this, provided no implication be made that the merely psychological unity is all that we have to deal with. I mean that Aesthetic as a science based on, and limited at every point by, psychology, is nevertheless distinct therefrom, and can only interpret its data in its own way. There is no reason why the psychological unity of attitude should not appear for Aesthetic still as a unity; and I should contend that so far as our experience goes we are fully justified in applying the term art, to such different things as *e.g.*, music, painting, poetry, etc. In this sense, art is, or may be, a real thing, whose reality we must represent to ourselves as something over and above the merely psychological process through which it is experienced; or, more strictly, there is something whose presence within the psychological process is to be explained and accounted for by other than purely psychological laws; and demands the recognition of aesthetic principles. It is largely for psychological purposes, however, that I take for granted the unity of art. If there is a unity, the only secure method would be to test every conclusion by reference to every type of art one could think of. Only thus could we hope to eliminate untrue generalization. As it is, the scope of illustration must seem unduly

restricted, more especially when dealing with the first view of art I want to discuss. The nature of the relation between art and religion may be something indefinitely broader or more comprehensive or more subtle than is implied by the types of art I have in mind, *e.g.*, very early Renaissance painting in Italy, or German music of the XVIIth and XVIIIth centuries. Nevertheless in the short space at my disposal I have to try to outline that relation with as much definiteness as possible.

I propose, then, to contrast two *prima facie* opposed theories of art. I shall first explain them, as opposed, at the risk, perhaps, of a little overstatement of both. Then I shall try to bring them together, and show that there is something in each which is applicable to all art worthy of the name. The real question is to decide where the balance lies. It is no doubt one of emphasis.

It is first of all quite certain that religion whether understood in a very narrow sense, or in a sense much wider than the generally accepted one, has at different times in the world's history exercised an enormous compulsion over art, compulsion, I mean as a motive power. That is to say, we have here to deal with those artists, who, as religious considered their religion, the claims which it imposed and the ideas or the ethic which it interpreted, as the most real fact of life, its claims the most pressing, and its doctrine the supreme interests. Art—if indeed they had that word in their vocabularly in any sense remotely akin to that in which we now employ it—art in itself did not enter into competition with those higher religious instincts. They forgot the claim of their art, if indeed the question of claim ever did come up. The very idea of rival claims set up by beauty was unreal. It was religion that had discovered art and brought it to birth. It was the concentration of attention and desire upon the supreme objects of religion that enlisted to their service everything that could be of use.

The value of a subsidiary thing, like art, was constituted by its power to adorn or interpret these; not as something worth following out for its own sake. As one of the ablest living exponents of the view let us take Alessandro Della Seta, whose book, *Religion and Art*,[1] states with extraordinary definiteness the case for religion as the uniqué inspiration and driving power of art. The development, for example, of the technique of perspective in painting or sculpture relief, takes place with a facility and completeness when a religion and an art grow up together that is not manifested under other circumstances. The motivation of art by religion goes for precision, delicacy and adaptability of technique as well as for spiritual power. But, of course, emphasis is placed upon the latter. Della Seta summarises his general conclusion in the words: "Man would never have set himself the task of representing men because of the beauty and nobility of their form. The form of men appeared beautiful and noble because it had served to clothe the Gods. Man therefore possessed art because he had religion."

Readers of Ruskin will at once perceive the close affinity of Della Seta's Aesthetic with his. Now, though, I should wish to emphasize what may appear to many paradoxical, *viz.* the thoroughgoing consistency of Ruskin's Aesthetic, I think the difficulties in this first theory with which we have to deal will become inevitably apparent if we approach them through Ruskin. The crux of his difficulty appears in his handling of the painters of the late Italian Renaissance, say, Titian and Michaelangelo. His position indeed with regard to the latter is rather curiously ambiguous. But at all events, he feels elements in Angelo's genius which he cannot justify on the principles which he would like to use. He is aware of his power and his pride, does not like them from the religious point of view, yet is baffled by them as somehow presenting very great aesthetic claims. So he

[1]Alessandro Della Seta, *Religion and Art*, English translation, 1914.

summarises Angelo in the rather remarkable phrase, no doubt, profoundly true—"He was proud, but not proud enough to be at peace." Titian exhibits a more complete bankruptcy of, at all events, the application of Ruskin's principles, if not of those principles themselves. It is true that for Ruskin as much as for any man some of Titian's greatest pictures are motived by religion, but there are others whose aesthetic value is in no way explicable, through the religious or ethical motive, or to be substantiated that way. And of this Ruskin eventually became aware. Perhaps it is possible, through Della Seta, to bring the difficulty to a clearer focus. The Italian Archeologist presents us with material that is of absorbing archeological interest and of enormous religious importance. But this material—is it all artistic of the quality it is claimed to be; can its value be made good as purely aesthetic? This, frankly, is what I doubt, and I should like to point out one very significant thing: that both Ruskin and Della Seta make some of their points by completely ignoring the fact of music in Reformation and post-Reformation times.

And now for the statement of the opposite view—that art as creating something that is of value in and for itself is an object that may and ought to be pursued just for the value of the beauty that it produces. The artist is here liberated from any other claim. If religious symbols offer him something which he can incarnate in paint—or other medium—and thereby help him to get the purely aesthetic quality of which he is in search, well and good. Let him use it. But the governing aim is obviously not the religious one but the aesthetic. Further, the aesthetic value is one which sets up a thoroughgoing rivalry with other values. Perhaps there is nothing in life of such worth as art. If so, there is an obligation—for the artist, of course—to pursue it at all costs. The ethical and the religious, in the old sense, becomes subsidiary. But—and this is all important

—religion is not thereby ruled out of life. The true religious instinct is that which is able to place meaner aims in subjection, and which, recognizing the supreme claim of art, makes sure of that, spends the life energy upon that, or rather sacrifices every other mode of activity for that.[2] Of course, the ethical claim, too, may be fully recognized, and placed side by side with the aesthetic, as equally the object of the religious instinct. But I think we must be prepared to consider the view in which the aesthetic has the supreme claim, and in which the ethical, though not ruled out, is made subordinate to the aesthetic.

I propose now to consider the analogy between the dream —I mean, the literal dream of sleep—and any work of art, or rather, of the psychological process of its creation or of its appreciative enjoyment by the men and women to whom it is presented. I believe myself that this analogy is very far reaching, and that its elaboration and application with all due criticism and care will be one of the most powerful methods of aesthetic in the near future. In applying it to our present problem we have also to consider the analogy in relation to the social art—consciousness as a whole, besides individual works of art. The general principle of the dream in relation to our ordinary consciousness, can, I think be clearly enough stated so far as concerns our present purpose. In the dream we have presented to us a succession of pictures or episodes which are to our waking consciousness *prima facie* unintelligible. They seem confused, bizarre and silly. In reality, and on discovery of the fact that the whole of the imagery of the dream is symbolical in character, they are not unintelligible; but are on the contrary pregnant with meaning. They are exact, subtle, and what is most relevant to our purpose now, of great importance to us.

[2] *Cf.* Clive Bell: *Art.*

But the interpretation of the dream, the process by which we become conscious of ourselves, of the dreaming part of ourselves, is as difficult, as it is felt to be wonderful when we begin to take the first steps in it. As well as understanding the dream, we find it take on a new aspect. That is to say, the process of becoming conscious involves, among many other changes, a change in the aesthetic quality of the dream itself. It is difficult I am sure, to convince anyone of this who has not begun to realize the true nature of his dreams; one must really experience for oneself the extraordinary deepening of the whole emotional tone that may follow upon grasping the significance of even a single symbol in a dream. Everything is changed, everything becomes illumined, everything in the dream responds to the new point of view; it is like the resonance of the whole mind to the new and correctly pitched note.

The first point, then, to which I wish to draw attention in the work of art as explicable through this analogy with the dream, is that we find the same process of becoming conscious correlated with an interpretation, or a re-interpretation of the symbols employed by the artist. Now this process of becoming conscious may occur within the artist himself, or during the appreciative enjoyment of the picture or the music on the part of someone quite remote from the artist, or it may be the character which marks the phase of transition from one kind of social art consciousness to another. It will be simplest, however, to illustrate this transitional phase of consciousness or of becoming conscious through the individual artist. I shall take as an example Goethe's extremely familiar little lyric, *Wandrers Nachtlied*. The impression the poem has made on myself at different times tallies exactly with the episode related of Goethe, but of course others may feel differently. At the same time, there is absolutely no doubt in my own mind that this is actually what happens in our reading of poetry

and in our enjoyment of all kinds of beauty. The episode is as follows: In the Autumn, 1780, Goethe spent the night at a shooting box on the summit of the highest hill in the neighborhood of Ilmenau. There he wrote the familiar lines

"Uber allen Gipfeln ist Ruh,"

"Over all the hills is rest; over all the trees thou markest scarce a breath. The little birds are silent in the wood. Do thou but wait; for soon shalt thou too be at rest." They were written in pencil on the wall on one of the upper rooms. There was no conscious expression, then, of a desire for the final rest. It was only a simple lullaby. About fifty years later, in 1831, the year before his death, he re-visited the house, went upstairs, searched for the poem on the wall and amid much emotion, repeated the last two lines, "Do thou but wait; soon shalt thou too be at rest."

I repeat that it seems to me quite natural for any of us to read the poem in these two ways. There is the acceptance of its apparent meaning, an impression of simple beauty, and there is the impression that comes through the re-interpretation of its symbolism, beauty of a much greater and profounder kind. The first point, then, that I wish to make is that we must admit the place and claim of the art that remains in its own unconsciousness. Symbols spring up from the unconscious of the artist—or perhaps, quite as well—from the deep social unconscious of his time, and shape themselves in aesthetically pleasing form in the picture. But here emerges a point of difference between the dream and the work of art. Dreams are not necessarily, though some dreams may be, aesthetically pleasing; art must be so, or else it is not the matter with which we are dealing. Now, in virtue of this fact, I suggest, though I am by no means sure, that there is always present in the creation or enjoyment of art at least some trace of the transitional phase, that I have described as the becoming

conscious. The unconsciousness of art and of the dream are never quite of the same kind; their correlation seems to involve an element in the former that is not, or need not be, in the dream.

That does not alter the fact that there is some art which shows, relatively, a profound unconsciousness. I have now to indicate the connection of such unconsciousness with the type of art which forms the ideal of the first of the theories we have been discussing.

When a great artistic movement seems to owe its power to the driving force of religion in a manner generally indicated by this theory of art, it is clear that the main religious ideas, say of God, of Immortality, or the death and resurrection of Christ must be held in complete conviction. The whole of the symbolism which accompanies them and through which they are stabilized and made accessible to faith comes to the mind as real. The intercession, for example, of the Mother of God is not symbolical in the sense that it stands for some other reality imperfectly apprehended. This intercession is itself believed in literally as a fact. Our difficulty here, of course, is that even in the great primitive periods of art we shall be encumbered with an enormous amount of conventional work, or even of work totally indifferent to religious conviction. But that hardly alters the main line of our argument. We do recognize, and I suppose, art critics of all sorts and conditions of views will be agreed in recognizing, the primitive type of work that bears the peculiar artistic quality I am trying to indicate. The picture—or whatever it may be—is laid before the people as actually representing what happens or exists. The artist believes it and they believe it. But now observe the limitation of this type of work—rather, I would say, observe the extraordinary transformation it undergoes when subjected to the scrutiny of people whose art consciousness has undergone a total change. It does not cease

to be beautiful. Perhaps it becomes more beautiful. I fancy that is true of many pictures. At all events, it is changed aesthetically. The intercession of the virgin is no longer the most important symbol of religious life, or the fact of intercession is denied. If we feel totally indifferent to the virgin, the picture may be nothing more than a somewhat strange, somewhat beautiful, somewhat perplexing piece of imagery. If it was at any time, or for any appreciative critic, a good picture, it will not likely ever be totally indifferent to anyone who likes pictures. But to us of the present generation who begin to wonder very much what the intercession of the virgin really meant, it will have all the added wonder of a half-interpreted dream. But that is not my point. Interest and wonder are good things. But they are not enough here. What I feel certain of is the deepening of the purely aesthetic appreciation of pictures of a bygone age with our deepening consciousness of the psychic forces at work in epochs of history and levels of social evolution which our general cultural resources are only now enabling us to understand.

But now let us go back to the artist and the people who looked at his pictures during his own life time with what special resources they had at their command and under the special limitations their age imposed upon them.

The artist as believing the liberal intercession of the virgin was unconscious of the nature of his own soul and of the forces at work in the social consciousness generally—unconscious, that is to say in a sense in which the art lover of a later time is not unconscious of these things. Now here I must try to distinguish two points of view—to distinguish them first, but really in the hope of showing that they may not be so different as I state them. It might be a real intellectual gain for the artist to become conscious of the hidden forces thus operating within his art instinct while yet it might be, aesthetically, either hurtful or in-

different. The point of view therefore of which we have to make sure is defined in the question whether or not it be of *aesthetic* value for the artist to interpret his own work in the sense of coming to a deeper consciousness of the symbols he employs. I contend, first, that it is in any case of aesthetic relevance and import whether he do or do not so become conscious; and in the second place, that art never really comes to its own until in some measure at all events it shows this increased consciousness. Or, in other words, I say that one of the great contributions that art makes to human life is in its interpretation of those marvellous symbols that our unconsciousness whether individual or social is continually bringing up to us. But this contribution may be intellectual, or economic, or what-not. Well and good so far as it goes. I should not therefore be content until I could prove that this contribution has also its purely aesthetic value. That is to say, that art locked up in its unconsciousness does provide us with beautiful things; but that the beauty provided by art which is conscious of itself and can by genuinely artistic means offer the interpretation of its own symbols, is of a higher order; and further, that this phase of transition, of heightening and deepening consciousness, must be one of profound importance for any attempt to say what the nature of art really is and what its value is alongside other values in human life.

These considerations should help us not perhaps to state, but to indicate, the limitations which the ethical or religious view of art imposes on it. So long as the reality of the symbols belonging to primitive religious faith is a matter of absolute conviction, there may be enormous driving power available for art production, but such conviction sets correspondingly strong forces at work which operate against the emergence of that particular character of art in virtue of which it can become its own interpreter. What, then by way of corrective or supplement, has the other

theory of art to give—that art, in its own right, has a place in life, and that it is at least legitimate for the artist to work in the full consciousness of the pure aesthetic value as an end to be realized?

First it is only in the full consciousness of this purely aesthetic purpose of art that we can entirely free ourselves from the confusion produced by the presence in it of qualities, in themselves excellent, but possessing a value which is not purely aesthetic. We do not, of course, know, we can at best dimly feel, what constitutes excellence in art *qua* aesthetic. It may involve things ethical, religious, economic, historical; or it may not. The excellence of different forms of art, music, architecture, poetry, etc., may involve these things in different degrees; or it may not. But it is profoundly important that we should not seek in art, for aesthetically extraneous qualities; or allow a bias in ourselves in favor of certain things we like which are not wholly in subserviance to beauty. To admit that it is sometimes possible, or that it is in some way possible, to be conscious of a strictly aesthetic purpose in art will perhaps be one of the things that may help us to this impartiality of appreciation. But further, such a view is most useful as taking up an attitude of uncompromising criticism towards those societies, or epochs of social development, in which religion has indeed been powerfully operative, but has been unproductive towards art. In the claim made by this theory for the fundamentally religious character of attention or devotion to art itself as the aesthetic end, it is formulating a doctrine that human life cannot afford to do without. Granted that the aesthetic is either the greatest value, or one of the great individual self-substantiating values exerting claim upon human experience, the ministry of art becomes a correspondingly urgent religious function.

This is, no doubt, a very curious inversion of the accepted relation. And it is just in this relation that some-

thing will have to be said in criticism of the whole view and of the attitude which it implies in the artist. Religion it may seem is somehow wrong placed with respect to art. It ought to be the channel of nourishment for art, to provide its contact with the earth—with the deep-lying, universal instincts of our nature, with what has been called the collective unconscious. Let us examine, then, the nature of this possible defect in the light of what help we can get from the psychology of the unconscious.

When an artist works under what I have called the compelling power of religion, he does not set out as it were wondering to himself what symbols he shall use. Rather they are presented to him. And when he is confronted with, say, symbols of sacrifice or submission, cast up before him as it were by the turbid waves of his unconsciousness, he has got to accept them. If he refuses them, or eludes them, his whole inspiration lapses. His strength as artist depends upon his allowing himself to be carried along by the flow of his own individual psychology, a power which he does not know and which, in its main current, he does not attempt to control. By this means the artistic handling of his problem has the stability given by the convergence of all the lines of symbolism which belong to his unconsciousness from his birth, or even from his pre-natal existence. They all concentrate upon that problem and he has behind him the power of their united and convergent action. It might be possible to illustrate this in a hundred ways from Dante alone. On the other hand, the artist who has freed himself from this compulsion exerted by religion, and who is out consciously upon the search for discovery of beautiful from alone, may choose from among the various symbols presented. But in virtue of the exercise of this choice, he is liable at any moment to step out of the main primitive currents of his psychology. If he does so, the stability of his attitude to the problem is gone. What happens is that

he merely dips down into the unconscious, and brings up this and that isolated symbol. His very consciousness of his aim is something that defeats itself. For a symbol by itself is little or nothing. It is its psychological setting alone that gives its significance and its power. That is why so much modern art seems to be of the nature of the play of little bizarre phantasies and curious forms—that please, no doubt, yet which are felt to be strange and bewildering as much as pleasant. And yet the movement is full of significance. The artist liberates himself to anything that comes from the unconscious, to anything that seems beautiful. For the world of the unconscious is of enormous interest, and the conviction of the artist is always that it is the very stuff out of which beauty is to emerge, the raw material of beauty. Let us give it free play in art; set it to work to amuse us, and let us entertain ourselves watching its bizarre freaks. They are instinctive, natural. They have the right to expression. Nothing else will furnish the true forms of art. Anything else is too intellectual, too sophisticated. And so the extreme of consciousness goes out upon the search for the extreme of unconsciousness. Nor may we justly object. Besides, the very oddness and intractability of those bizarre phantasies involve the discovery and invention of new art forms and new methods.

There are many of our dreams that arouse our interest, but are so extremely odd that is difficult to communicate them by any ordinary means. When you are talking to a lady who is looking straight at you, you cannot, under the conditions of time and space, see the back of her. But you can do this well enough in a dream. Such a dream is difficult to render aesthetically in painting, still more so, perhaps in music. Nevertheless, to confine ourselves for the moment, to the simpler problems of the two, an attempt could be made by the former art to render the characteristic

peculiarity of the dream. Such, I take it, is the explanation for example, of cubism in modern painting, say Picasso's picture of the *Lady in the Armchair*. This is probably nothing else than an artistic device through which our unconsciousness can disport itself with a little more freedom, and admire, at one and the same time, aspects of the lady that under the limitations of our waking consciousness cannot so be enjoyed. The picture takes on the fascination and mystery of the dream, with not a little of its fear. The lady is veritably alive with motion—motion, inscrutable, formidable, concentrated—as in the lightning flicker of her fan. Most dangerous—as if yielding unwarily to a seductive vertigo, you should tumble forward, be whirled round, and your universe blacken and founder in an *eldrich scraich*.

We cannot on the whole, treat altogether lightly this effort at a direct approach to beauty, or the conscious attempt on the part of the artist to avail himself immediately of the resources of the unconscious. It is only a guiding principle that is absent, though that indeed may be difficult to discover. The problem of art would therefore be to obtain a full consciousness of its aim and value as purely aesthetic while yet the artist should not lose the driving power that seems to come only from a conviction of the reality of other values, not aesthetic, or at least not necessarily aesthetic. So stated, this seems little other than a paradox. Yet as a matter of historical fact, we should no doubt find all degrees of relative emphasis. I am myself inclined to the view that the great times of art in the world's history belong to the periods of not quite complete liberation from primitive convictions. The crest of the wave of such a movement is perhaps what brings about the final liberation. That is why the period of perfect liberation seems often enough to be coincident with the emptiness and desolation of the spirit. But in this regard history seems to teach a clear enough lesson. The greatest art

movements are wave-like in character. The crest is magnificent, and catastrophic. In the period immediately succeeding the decline, it has always been the habit of people to bewail the fall and to look about helplessly for some continuation of the art-life along the old lines. This seems totally wrong. What history gives is the revivification of the undying instinct to beauty—but in some totally new and different form—a form beyond the scope of any prophecy.

This character of climax seems, therefore, to belong to art. Where you find magnificent art, make up your mind for the catastrophe. In fact, this seems as inevitable, and as natural, as for a man to awake out of sleep. That there is a fundamental aesthetic value in the coming to consciousness by art of the meaning of its own symbols is not incompatible with the probability that this act or transitional phase is something that eventually strains it, as art, to breaking point; and therefore banishes, for the time being all possibility of the value in question. Yet this straining does not seem capable of resistence or check. Perhaps it is like birth, and its contribution to life greater than one might suspect.

At least, this is how I interpret the work of an artist like Beethoven. It is not that he gives us Wagner. Wagner is excellent, but he does not explain or validate Beethoven. What Beethoven gave to Europe was liberation. Europe needed liberation more than anything else, and perhaps this liberation that he gave was for the time being incompatible with the soaring music of the Bach-Mozart-Beethoven type.

One word more. We have need, we have discovered, of our dreams, and of all that the unconscious can give. But in the cold day-light of awakening we are not really cut off from that unconsciousness. We have access to it, and we can utilize it. If it is part of the excellence of art itself that

at a certain stage it comes to a sudden end, there is a relation to establish between the reflective mood thus deprived of power artistically to express itself and the foregoing period of more emotional or artistic life. The problem of the relation of art to reflection and to other non-aesthetic spheres of culture and of life has hardly yet been entered upon. Yet it is one of almost infinite scope and possibility.

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CRITICISMS AND DISCUSSIONS.

CROSS-PURPOSES IN AESTHETIC THEORY.

The recent symposium on "Mind and Medium in Art," printed in the British Journal of Psychology for October, 1920, affords a capital occasion for appraising the present status of aesthetics. There is surely need for such appraisal; and though a short summary like this must omit many points of interest and value, it may, by putting together both the common ground and the chief divergences adduced by the five contributors, indicate the main directions in which a sound aesthetic theory ought to proceed and the main perils which it ought to avoid.

The first two disputants advance views which bring before us a sharp opposition. Mr. Charles Marriott, who opens the discussion, contends that distinctions between the several arts should be based solely on the medium—"the tools and materials"—employed, not on any such criterion as reproduction or non-reproduction of nature, and that such distinctions cannot soundly be drawn *within* any art. Hence aesthetic enjoyment is absolutely dependent on a recognition of successful use of the medium. "If it be objected that the reason is practical, my answer is that practical and aesthetic reasons are at bottom the same thing; or, to put it another way, that aesthetic appreciation is dependent on the sense, which may not be conscious, of practical problems effectively solved" (p. 2). And again, "The objection that we can enjoy works of art without knowing how they are done, and that in moments of highest aesthetic enjoyment we become unconscious of the means, is only apparent. Directly the means are used out of character we *do* become conscious of them, whether we understand their characteristic use or not" (p. 4). Hence the conclusion that the medium employed furnishes the sole valid basis of discrimination, and that the time-honored distinction between arts and crafts is illusory.

To this bold pronouncement, Mr. A. B. Walkley opposes a more traditional doctrine. He looks with suspicion on this emphasis on

the external, and bids us turn our gaze inward. "The inquiry from without having failed to bring us to the goal, may we not have better luck in trying the method of inquiry from within? After all, that is the normal method. The history of aesthetics through the ages runs mainly along that line. Aestheticians, from Plato onwards, have nearly all agreed in supposing that art is primarily a 'function,' as the mathematicians say, of the artist's mind, rather than of his medium or of his tools" (p. 11).

In this sharp conflict of two one-sided views, it is evident that the second offers rather less hope of profitable results than the first. Aestheticians "from Plato onwards" have abounded in mistaken judgments just because they have insisted on inquiring from within, instead of ascertaining, as widely as possible, the facts of actual artistic accomplishment. Art was an element in human life long before explicit philosophizing, let alone aesthetic theorizing, began; and it is at least likely that a direct, even if hasty, approach to actual conditions will yield more profit than will a premature attempt to draw aesthetic conclusions from a general philosophical position, before the more special field has been surveyed. Aestheticians have too often been prone to shelter themselves behind great names, and to ignore facts which could not conveniently be brought under the accepted formulas.

It is just this error of premature conclusions that the fourth and fifth[1] contributors to the discussion avoid. Mr. Edward Bullough lays out the aesthetic field with gratifying comprehensiveness. "We would, I believe, all agree that art represents but a segment of the whole range of aesthetic activity. The aesthetic attitude embraces the appreciation of natural objects, as well as of artefacts; it covers certain borderlands of art, such as ceremonial and ritual, which have admittedly aesthetic affinities; it permeates ordinary human intercourse in what we vaguely call 'manners.' As a matter of personal conviction, I hold that there is nothing in the whole range of personal experience, from sheer sense-experience to the most abstract thought, which may not be the objects of aesthetic contemplation" (p. 27). From this hopeful beginning, he goes on to develop a theory of the static and the dynamic aspects of art, the former denoting art-objects as external things, the latter their relation to the creative artist or the actively appreciating recipient; to point out

[1] Exigencies of space compel me largely to pass over Dr. Watt's paper, which is more concerned with special topics than with a general survey, and is less easily summarized.

the corresponding errors of treating the artist *in vacuo* and of over-emphasizing the social aspect of art; to discuss the correlation of medium and experiment; to analyze the idea of technique; and to sketch a theory of the imagination. We have exchanged the sharp conflict of isolated views for a broad and inclusive outlook, with a gain in justness of perception that is manifest throughout Mr. Bullough's paper.

The same spirit informs the paper by Professor C. W. Valentine which closes the discussion. He, too, has a pair of errors to note in much previous theorizing. "The best known theories of the beautiful seem to err in the following respects: they take one aspect of the aesthetic experience, or one or more characteristics of certain beautiful objects, give quite convincing illustrations of these isolated factors, and then generalize as to beauty or the aesthetic experience. Very varied theories can easily arise thus; for so complex are aesthetic experiences that many varied elements may have a share in such experiences. Still more readily can we generalize if we limit our field of discussion to one of the arts" (p. 47). And again, "The whole question as to what is beauty, or what is the aesthetic experience, has been unnecessarily complicated, in my opinion, by the undue attention given to the activity of the artist. If it be granted that we can have a truly aesthetic experience without ourselves creating (externally) an object of beauty, it seems unnecessary to introduce at first a discussion of the artist's creative activity in our search for our theory of the beautiful. At least, it gives us an additional problem to solve, and in facing it there is usually a failure to recognize the extreme complexity of the mental processes involved" (p. 49).

If we now look about for the common ground on which this lively interchange goes forward, we find one portion of it in the emphasis on the concrete work of art as the only safe point of departure. "It seems perfectly clear and certain," says Dr. Watt, "that without a basis of sensory beauty there can be no work of art. Pure intuition or not, a work of art must be created, it must be embodied, and beauty must permeate its whole being, sensory, perceptual, imaginative, from whichever end of the series its soul may spring" (p. 20). It is the merit of Mr. Marriott's view that it emphasizes this; it is its defect that it emphasizes the aspect of technique too exclusively, so that, as Mr. Valentine points out, his dictum as to the importance of the medium has greater validity in its negative form.

The more unfamiliar the medium, the more conscious we are of the technique as a separable element; though in any case, defective use of the medium is always detrimental to aesthetic satisfaction.

But to lay too much stress on the importance of the medium is clearly unwise. Indeed, we can scarcely avoid asking why, if differences of medium are so important as to be the constituent principle of the division between the arts, there may not be other differences of corresponding importance?—such as differences of temperament, attitude, intention, on the part of individual artists. These, too, might furnish an occasion for fresh groupings. On general principles, it is unlikely that the separate arts should be irreducible units; nor does the failure of any single mode of analysis prove them to be such units. Because the differences as currently drawn are confused or hard to apply, it does not follow that they are negligible; they may but need closer examination, and clarifying. Indeed, any term which has been long and widely used must apply to *something;* and it surely is better to see just what degree of correspondence to truth and of application to actuality it possesses, instead of flatly discarding it.

A second point is the sense of the complexity of the aesthetic field and of its phenomena, of all which art in the stricter sense is but a segment. We, therefore, need a specific name to designate this special province; as a provisional measure, we might restrict the name "aesthetic theory" to it. At all events, it occupies its own place, and many of the results of experimental aesthetics apply to it, but remotely or partially. In it, however, we have the aesthetic experience in its most highly developed and concentrated form, and hence use it as the field in which aesthetic qualities are best displayed, and the theoretic study of them is most richly rewarded. But to disentangle complexities aright requires us to keep our modes of approach as distinct as we can.

This field of fully developed artistic experience offers us the triad of work, artist, and recipient, each with its contribution to the total complexity, and each with its special set of dangers attendant on over-simple conceptions. The importance of the individual work has already been brought out; but the problem of the artist, though, as Mr. Valentine says, it may wisely be postponed, can hardly be left out of account. The problem is admittedly difficult; we may reconstruct an artist's probable attitude on insufficient or faulty evidence, and even his own testimony as to method and intention may not be wholly reliable. An artist may not be conscious of all that he has

put into his work, and himself be unable to account for the value which it ultimately assumes; and conversely, he may have done all that he intended to do, and still have accomplished nothing of permanent worth. And lastly, since works of art are not produced in a vacuum, there is the problem of the recipient; the qualities of sensitiveness, response, flexibility, range of interest, which he should have, the defects of dullness, inertia, rigidity, narrowness, of which he should rid himself; not to mention the extraneous influences of fashion, prejudice, and the like, which must be removed or allowed for.

Here are surely questions enough to provide discussion for a good while; but even if, in the interest of manageability, we set aside those raised by artist and recipient, the work itself remains if anything does, and is often our only basis for surmise as to producer or reception. We know practically nothing of the specific mentality of Egyptian craftsmen or Greek sculptors; they may not even be identifiable as individuals, but we derive very decided impressions from their works, and indeed, use them as starting-points for whatever we infer as to the artists' personalities. And what enables us to do this, if not the fact that the works were the outcome and embodiment of their creators' experience, now, by their survival, able to be adopted and incorporated into our own experience today?

If art thus mirrors experience, it is easy to account for the many varying views which have been held concerning its value and significance. There will be as much disagreement about them as about the corresponding traits of that general experience which art reflects. Temperamental differences will play the same part in aesthetics that they do in philosophy at large; what is vital to one observer will not always seem so to another, and such disagreements, while mankind retains its present constitution, will never be settled by argument. Despite the incidental optimism of several of the contributors to the symposium, there are very few points on which "all" will agree; and I myself have no illusions that the common ground thus far brought out will prove to have universal acceptability. It is enough if it commends itself to a sufficient number to make its occupation really fruitful—as indeed I believe it will.

At the same time, it is easy to exaggerate the amount of conflict which prevails both in individual aesthetic judgments and in general aesthetic theory. Dr. H. Wildon Carr has darkly alluded to "inextricable confusion," "widest divergence in the varieties of aesthetic theory," and has even gone so far as to say, "No two individuals

whatever degree of culture they possess, seem able to be of one accord in their aesthetic judgments, nor does anyone seem able to preserve his own aesthetic judgment invariable." But after all, are such divergences really wider in aesthetics than in philosophy as a whole? I suspect that the reason that they seem so is that aesthetic views enter more extensively into ordinary discourse than do purely metaphysical positions; but surely the latter, as at present held, diverge widely enough to satisfy anyone. It is, perhaps, significant that Dr. Carr makes his sad discovery in the course of an examination[2] of a system which has indeed done much to introduce confusion into current discussions, and which is the direct antithesis of the position developed in this paper—the system of Croce.

Here we have neither time nor space for any real examination of Croce's doctrine; I need only say that I fully endorse the doubts expressed by Messrs. Bullough and Valentine. Like the former, I am uncertain "in particular, about the nature of the 'intuition,' the essential 'lyricism' of art, and their consequences." It is not merely that Croce's positions often seem in direct conflict with ordinary experience, but that they present notions which neither convey meaning in themselves nor help in solving aesthetic problems. What, for instance, shall we say of the dictum that "if we think of man at the first instant that the theoretic life is disclosed to him, his mind still unburdened by any abstraction or reflection, he in that first instant, purely intuitive, could be nothing but a poet"?[3] Does this hypothetical phantom resemble any poet we have ever heard of, or represent anything which the analysis of experience actually discloses? Or what shall we say of this: "The work of art is a spiritual fact, and therefore is never external (physical). A statue of a series of tones may be weighed and measured and counted by the physicists, who are wholly indifferent to the spiritual significance of what, to them, are external things. But for the aesthetician there exist no things which are measured, weighed, or counted; there exist only images, spiritual acts. To find a passage or a connection between the spirituality of the image and those physical complexes of colors, sounds, and voices is a desperate task."[4] Is the difference between bronze and marble relief sculpture a difference solely of "measuring, weighing, and counting"? Does the difference between a sequence of notes played on a flute and the same sequence played on an oboe consti-

[2] *The Philosophy of Benedetto Croce*, pp. 155, 153.
[3] *Problemi d'Estetica*, p. 15.
[4] *Ibid.*, p. 468.

tute exclusively either an immaterial image or a mere problem in acoustics? A theory of aesthetics which leaves such questions out of account is surely an oddity; and indeed Croce's whole system seems to me an extraordinarily persistent attempt to impose a set of water-tight intellectual compartments on the broad current of experience. One might even go so far as to maintain, in exact opposition to Croce, that we can conceive, say, of a physics of literature, a chemistry of literature, a psychology of literature, and a metaphysics of literature—that is to say, of the points of view of physics, chemistry, psychology, and metaphysics applied to the experience reflected in literature in a manner analogous to their application to what experience directly furnishes. This may be extreme; yet, I am not sure that it goes so far in its direction of error as Croce does in his.

But to return from the Crocean problem to our previous standpoint: it may be remarked that the latter makes it necessary to determine how close and how extensive is the contact of any particular art with experience, and whether its presentation of it is direct or indirect. Literature is obviously, of all the arts, that in which the contact is at once closest and most varied, to such a degree that under our present system of education we are prone to interpret other arts in terms of it. Yet in ages which were less subject to the tyranny of the written word, painting and sculpture conveyed many ideas which we now receive through the medium of literature. At the other extreme stands music, which has at least in modern times, been perhaps the most specialized of the arts, and certainly the one least susceptible of an informative application. As for the immediacy with which the arts present experience, literature again occupies the most intimate position, in an age when knowledge of reading and writing is so widely diffused; painting and sculpture can be directly seen, though not necessarily grasped in plastic terms; whereas music and drama by the very law of their being, need an intermediary. It is conceivable that a race of beings might exist for whom musical sounds were as direct a medium of intercourse as speech and writing are for us; but we are not such a race, and even the trained musician does not long forego the translation of written music into audible. In this field Mr. Marriott's principle of the value of technique has obvious and extensive application.

If, then, art is the embodiment of experience, nothing that experience offers is necessarily alien to it. Yet, in effecting that embodiment, it will choose with an aim which is not identical with that of

the philosopher, or the scientist, or the man of action, and it will bestow on the selection and handling of the medium a care which they need not always employ; if indeed the medium be not one of which they can make no special use. The aim of art will be, as Mr. Bullough has finely said, "Vision conceived in terms of the medium." It will not be a substitute for experience, but it will render permanent much that we should regret to lose, and the making of it permanent is itself a delight. "Even for the artist," says Mr. Valentine, "I can scarcely believe that the richest experience is always in the creative imagination; but rather that normally, especially in visual art and in music, the artist embodies his imaginations in a medium not only to communicate them to others, as Mr. Clutton Brock holds, but to make fuller and more intense his own aesthetic enjoyment. The testimony even of some poets goes to show that the impulse to write is dependent on a craving to enjoy more fully than can be done in mere imagination that which they cannot find in real life. 'If we had life,' wrote Wagner, 'we should have needed no art'" (p. 54).

These last words, however, somewhat overshoot the mark. Art is not merely an escape from the actual; its use is not "to give some shadow of satisfaction to the mind of man in those points wherein the nature of things doth deny it," as Bacon conceived, or to seek, as with Pater's Watteau, "after something in the world which is there in no satisfying measure, or not at all." If we *had* life, in the sense of absolute experience, we should not need art—nor philosophy, nor any other of the means by which we attempt to retain and unify our fragmentary and elusive existences. But, while we are as we are, we shall need art; and if we are to understand what we are about, we shall need a sound theory of art—a theory which gives due place to all the factors of the aesthetic result, without over-estimating any; which sees the artist in his environment, not in a social vacuum, but which exaggerates the importance neither of him nor of his surroundings; which appraises the sensuous no less than the mental aspects of the work of art; and which regards no element of the whole as too simple for careful discussion and analysis. It is precisely because the British symposium seems to offer so many bases for the establishment and progress of such a theory that I have ventured on this commentary, with the further hope that on this side also it may serve as a rallying-point.

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THE PRINCIPLES OF THE FUNCTIONAL CALCULUS.[1]

In the following pages, we shall endeavor to characterize a development of thought which assuredly constitutes the most important manifestation of contemporary mathematics. This development deserves to attract the attention, not only of geometricians, but also of those philosophers interested in the evolution of the principles of science. The theories to which the name of "functional calculus" has been given have been obtained by going back to the very origins of mathematical science, by thoroughly examining and generalizing the notions which form the basis of analysis itself: operation, function. Strictly speaking, the functional calculus does not constitute any particular theory with rigid boundaries. The somewhat vague denomination of "functional calculus," "calculus of functions" would include the whole of analysis; and indeed the functional calculus is essentially a generalization and an examination of the principles of classical analysis.

In accordance with a method we have already used, it is our intention to emphasize the essential ideas of this phase of the development of mathematics and to show their genesis. Our point of view is thus in marked contrast with that of an encyclopedia the main object of which is to sum up the greatest possible number of particular facts, without troubling too much about the way in which they are connected with one another.

The various streams of ideas we shall isolate from one another in order to comment upon them for the requirements of exposition are really interblended. There is always something arbitrary in the process of cutting the bonds which unite a notion with connected notions. The "mysterious unity" mentioned by Charles Hermite, which brings together "theories apparently farthest separated," loses its mysterious character when we remember that originally, and for the purposes of logical demonstration, we cut up real, complex, and synthetic notions into bits and, so to speak, work on these fragments of ideas. The task of genius will often be to bring together these fragments that had been separated.

We will take almost a commonplace example as an illustration. It has long been possible to regard algebra and geometry as two

[1] Authorized translation by Fred Rothwell, from the *Revue de Métaphysique et de Morale* of July, 1913, Vol, XXI. pp. 462-510 (condensed).

distinct sciences. Descartes, however, has shown that they are closely linked together, and pointed out the immense advantage to be obtained by combining the two.

Consequently we shall study separately theories which have sometimes developed independently of one another, and have sometimes interfered. For the reasons we have just explained, it will be almost impossible to make any hard and fast distinction between these different lines of investigation.

By a circumstance anything but fortuitous but rather inherent in the nature of things, the new theories were on the point of supplying mathematical physics with a powerful instrument of calculation. Indeed, to discover fruitful theoretical principles is the same as to grasp, for our calculations, directions which coincide with the structural lines of the real. The wonderful facility with which the integral equation of Fredholm applies to the problem of Dirichlet, a problem that typifies a large number of problems in physics—"elliptic problems"—is simply an illustration of this fact. Volterra, Hadamard, and other mathematicians utilized the new methods in various branches of mechanics and physics.[2] These numerous applications testify to the fecundity of the ideas; they constitute, as it were, the guarantee of their "objectivity." We shall not concern ourselves with these applications to physics; suffice it to mention that they are both numerous and important.

What is the origin of this *ensemble* of theories now called "the functional calculus"? What are the fundamental ideas of this mathematical doctrine? Taking solely the theoretical point of view and dealing with the matter in the way indicated, such are the questions we wish to answer.

With the above-mentioned reservations, we will trace back in three principal directions the phase of the development of mathmatics, the history of which we shall attempt briefly to describe. At the outset we shall study the generalization of the theory of operations; this generalization constitutes the formal part of the theory, but in our opinion it has had considerable influence in the formation of ideas of a more concrete character. Leibniz seems to have been the first to study the formal laws of operations (the resemblance between the formulas which give the development of the power of a binomial and the derivative of a product of two functions).

[2] See Heywood and Fréchet, *L'Equation de Fredholm et ses applications á la physique mathématique*, 1912, and Kneser, *Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik*, 1911.

These investigations, moreover, dealt with the philosopher's conceptions relative to formal logic. At the beginning of the nineteenth century, Servois endeavored to formulate the general properties of operations: commutativity, associativity, etc. Besides, as Pincherle has remarked, there is in Servois a certain confusion between *operation* and *function;* this confusion, however, might be called a happy one; for the development of thought we are describing was bound to end in regarding *operation* as an *object* (a kind of function): mention is made of the equality, the univocacy of operations, and also of their commutativity, their associativity, etc. This is the conception, in its generalized form, that has been reached by Pincherle and Bourlet. The latter, for instance, gives the name of "transmutations" to a very general class of operations which is defined as follows: we say we have defined a transmutation when we have enabled one or more functions to correspond to any function. Change of variable, derivation, integration are transmutations. Operation has become a functional correspondence.

The second stream of ideas which we shall study, one that has developed either along parallel lines with the first or by uniting with it, is of considerable importance. The ideas with which we are now about to deal are really the ideas at the very base of the infinitesimal calculus. Volterra, in his opening lecture at the Sorbonne in 1912, clearly showed that the theories which Fredholm, Hadamard, and himself have thought out the last few years, are connected with the elements of infinitesimal analysis, forming, so to speak, its necessary development. We should not follow up this development were we certain that the fine pages to which we allude were accessible to all readers.[3]

We have seen briefly how the concept of operation has become generalized. The idea of function has also undergone profound transformations since the time of Bernoulli, when, under the name of function, certain relations (algebraical, logarithmical trigonometrical) were said to bind a quantity y to a variable x. The integration of equations with partial derivatives of the second order led D'Alembert and more especially Fourier to a far wider conception of function than Dirichlet seems to have had, in its most general form, when he regarded uniform function as any *correspondence* whatsoever between a value of x and a value of y.

And so the idea of correspondence which constituted, as we

[3] *Revue du Mois*, 1912.

have seen, the fundamental nature of operation also forms the essential element of function. Having attained to the notion of correspondence or of relation, we seemed to have reached the most general conception of function. It was still possible, however, to generalize the concept of variable. This idea was necessarily inevitable. Function, indeed , is the unknown of functional equations (taking the expression in its broadest sense), i. e., the unknown of a differential equation, one with partial derivatives, integral equation, etc., as a number is the unknown of an algebraical equation. A *rapprochement* had to take place between the *numerical* variable and the *functional* variable. Volterra seems to be the first (about 1887) to have envisaged a function which depends on the *form of a line*, i. e., to have considered a function as dependent on a *continuous* infinity of unknowns. Henceforth the form of a line intervenes as an independent variable; a function depends on a continuous infinity of variables. In a daring generalization, Volterra extended to the functions of lines the fundamental notions of the infinitesimal calculus: continuity, derivative, differential, development in series of power, analyticity. Following the same line of thought, Hadamard considered functions depending on the form of another function (these are not functions of functions) which he has called *functional*. The calculus of Variations, which includes the whole of analytical mechanics, affords an instance of functions which come into this category, as, for instance, the integral which is to be made minimum depends on the form of a function.

Whereas the ideas just developed are essentially connected with the notion of continuous infinity and thus with the geometrical intuition of the continuous which is the foundation of geometry and the infinitesimal calculus, another orientation (the third stream of ideas we wished to mention) was about to appear, the one which, invoking the name of Weierstrass, we might call the arithmetical orientation. At the base of the arithmetical conception appears as the fundamental element the idea of a development in series, Taylor's series, Fourier's series, etc., a development the ensemble of whose coefficients has the same *power* (in Cantor's meaning) as the sequence of the natural integers. This was the point of view assumed by Le Roux, Hilbert, etc., when they considered that a function may be looked upon as dependent on an enumerable and therefore *discrete* (no longer continuous) infinity of independent variables: the coefficients of its development. The coefficients of a Fourier development, for instance, constitute the infinity of independent variables

on which depends the function represented by this development. In the language of geometry, a function may be considered as a point of space with an enumerable infinity of dimensions, each variable representing coordinate. Obedient to kindred ideas, moreover, Hill, Poincaré, and at a later date Helge von Koch, had already considered systems of a discrete infinity of linear equations with an enumerable infinity of variables. By conceiving of analysis as algebra wiith an infinity of variables, the concatenation of the ideas at the root of algebra with those that serve as a foundation for analysis is firmly established. In algebra indeed we have to determine a finite number of magnitudes by means of a finite number of equations; in analysis a function must be determined by means of a functional relation. As Hilbert remarks, a function may be regarded as known when the enumerable infinity of its coefficients is calculable: now, these coefficients may be determined by a *discrete* infinity of equations, and this system of an infinity of equations with an infinity of unknowns will be equivalent to the functional (differential, integral, etc.) equation. Thus we understand how analysis may be said to be the continuation of algebra. Hilbert endeavored to systematize the notions relating to functions with an enumerable infinity of variables: he dealt more particularly with quadratic and bilinear forms with an infinity of variables.

We have attempted to show that the functional calculus is at the meeting-point of a threefold development: the generalization of the formal properties of operations; the extension of the notions of the infinitesimal calculus, the fundamental idea of which, at the outset, is continuous infinity (the generalized function, in this case, depending on all the values of a function in an interval, a conception connected with the notion of the definite integral); and finally, the generalization of the arithmetical point of view, a function here being conceived as depending on an enumerable infinity of variables. To endeavor to establish, as has sometimes been attempted, that one of these last two conceptions is, speaking absolutely, superior to the other, appears to us meaningless, for we are confronted with two equally fundamental modes of inquiry. The one can be traced back to intuition and the geometrical continuous, the other is related to the primitive elements of arithmetic: to the natural sequence of numbers. The history of the theory of analytic functions teaches us, for instance, that geometricians, following their natural inclinations, have taken up the study of mathematical facts sometimes according to one conception, sometimes according to the other.

The reason why we have hitherto refrained from speaking of those new forms of equations called integral and integro-differential equations, is because we wished to make a clear distinction between fundamental ideas and the applications that can be made of them. The solution of integral integro-differential equations, as also the solution of differential equations or equations with partial derivatives, consists in the determination of a function in the ordinary sense. But the most satisfactory methods that have been decided upon for solving these equations are directly connected with the ideas we have set forth. It was with reference to a problem in mechanics that Abel first met with an integral equation, i. e., an equation in which the function to be determined is represented under an integral sign. Volterra and Fredholm, dominated by the fundamental conception according to which the unknown function depends on a continuous infinity of unknowns, have considered the integral equation as the limit case of a system of n linear algebraic equations with n unknowns, obtained by splitting the interval of integration into n parts;[5] the solution of the integral equation will be given by the limit of the solution of the system of equations: the continuous infinity of the solutions of the system will constitute the totality of the values of the unknown function in the interval of integration. In an integro-differential equation, the derivatives of the unknown functions are represented under the integral sign. The integro-differential equations of Hadamard approach the ordinary differential equations, those of Volterra approach the equations with partial derivatives of the second order, and he solves them by combining his method of passing to the limit with the methods used in solving equations with partial derivatives of the second order.

The equations with functional derivatives mainly envisaged by Hadamard and his pupils are different from the integral and integro-differential equations, because the unknown is no longer a function of a variable (or of a *finite* number of variables), but a function of lines. Whereas the integral and integro-differential equations may have appeared, and the integral equations actually have appeared, in the applications, previous to the development of the ideas we have just set forth, the equations with functional derivatives could only be envisaged when the functions of lines (or the "functionals") were known.

[4] P. Boutroux, *Revue de Métaphysique*, Nov., 1904.

[5] Volterra, *Leçons sur les équations intégrales et les équations intégro-différentielles*, Paris 1913.

We have seen how Volterra and Fredholm regard integral equations; their conceptions rest on a common basis, the consideration of function as dependent on a *continuous infinity* of unknowns comprised in an interval. Hilbert regards the function to be determined as a function of an *enumerable infinity* of unknowns which are the coefficients of its development in a generalized Fourier series and the integral equation will be replaced by an enumerable infinity of linear equation between an enumerable infinity of unknowns. The solution of the integral equation is indeed only a particular application of Hilbert's ideas. "Analysis with an infinity of variables," to use his expressions, constitutes a general method applicable to many problems.

Finally we will make a few remarks on the theory of functional *ensembles*, or rather, abstract *ensembles*, of which functional *ensembles* (the *ensemble* of continuous functions, etc.) form a particular instance. This study, with which Fréchet and some other writers have dealt, would constitute a kind of introduction to the functional calculus. In a logical *exposé* of ideas it is this theory—generalized—of *ensembles* which would compose the first chapter. As we said at the beginning, however, we have not adopted the point of view of logic but that of the historical development of notions. The reader now understands the philosophical range of works which can be traced directly back to the elementary principles of mathematical thought: to operation, function, continuous infinity, enumerable infinity.

I. THE PROPERTIES OF OPERATIONS.

If we wished to determine the origin of the theories we are now about to set forth, we should naturally have to go back to Leibniz. In the field of what is at present called the functional calculus, as in the theory of the differential calculus, this prophetic genius seems to have divined what was actually discovered at a later date. Pincherle, in the learned study of the *Encyclopédie des sciences mathématiques*[6] which he devotes to the functional calculus, considers that Leibniz's formula concerning the m^{th} derivitive of a product of functions constitutes one of the first interesting propositions to be connected with this branch of mathematics which has been called "calculus by symbols of operations:" There is a *formal* analogy between the development of the powers of the binomial and

[6] Pincherle, *Encyclopédie des sciences mathematiques*, Vol. II, 5, 1.

the operation of the derivation of a product of two functions Leibniz's observation, however, is not accidental in his work, it is connected with the conception which the philosopher formed of formal logic and of the differential notation.

The logical preoccupations of Leibniz[7] were blended with his reflections as a geometrician and a metaphysician. The conception of general logic and of a universal *characteristic* continually haunted his mind. The importance attached to symbolism by Leibniz is well known. Its *characteristic* was to be a kind of universal algebra. In calculus form this algebra would have given all possible relations between concepts. For reasons easy to guess, this integral symbolism could not be constructed. The mind, however, behind this attempt, naturally led Leibniz to study the formal laws of the calculus. It would appear that Leibniz had already outlined a sort of "general theory"[8] of the operations considered in their formal relations and their properties, that he had already the idea—quite a modern one—of regarding algebraical signs themselves as symbols of indeterminate operations." Leibniz distinguishes *similar* operations such as arithmetical addition and multiplication from *dissimilar* operations such as substraction, division, and the raising to powers. In the former, it is possible to invert the order of the terms without changing the result; in the latter this cannot be done.

More generally, Leibniz would appear to have conceived "the formal identity of really different operations, and the possibility of replacing them by one another in the calculus; especially when two symmetrical (*similar*) operations are superposed, such as addition and multiplication, they can be inverted without changing the result of the calculus." We thus see that "Leibniz's theorem" which by symbolic formula gives the m^{th} derivative of a product of functions is but the consequence of a general philosophical conception. It is interesting to note that the inventor of the differential notation was the promoter of the calculus by symbols.

But it was Servois who first studied systematically the formal laws of the calculus. He showed that there are certain very general formal laws which apply to objects as different, for instance, as a quantity and a symbol of operation; he has set forth the importance of the commutative and distributive properties, a knowledge of which now forms part of the first elements of the calculus.

[7] Brunschvicg, *Les Etapes de la philosophie mathématique*, p. 197.
[8] Couturat, *La Logique de Leibniz*, p. 303.
[9] Couturat, *loc. cit.*, p. 302.

To understand adequately the development of ideas, we consider it interesting to recall Servois's text.[10]

"Let

$$\phi(x+y+\ldots\ldots)=\phi(x)+\phi(y)+\ldots\ldots$$

"The functions which, like ϕ, are such that the function of the (algebraical) sum of any number of quantities is equal to the sum of the like functions of each of these quantities, will be called *distributive*....

"Let

$$f\,h\,z=h\,f\,z.$$

"The functions which like f and h are such that they give identical results, whatever be the order in which they are applied to the subject, will be called *commutative* with one another." After developments of calculis into which we do not wish to enter, Servios applies the foregoing principles to the functions he calls differential, which he represents by $\mathrm{E}(z)$, $\Delta\ (z)$, and $d(z)$, the *varied state*, the *finite difference*, the *differential*. Note that the *varied state* of the function $z=\phi(x,\ y....)$ is a function $\mathrm{E}(z)=\phi(x+\alpha,\ y+\beta,.....)$, the α and the β.....being the increases of x, y, etc., and that the *difference* of z is defined by the expression $\mathrm{E}(z)-z=\Delta(z)$. It is useless to recall the definition of the differential. "Now all the differential functions and their different orders, positive or negative, are *commutative* functions, both with one another and with the constant factors....In the preceding, we have sketched the *ensemble* of the laws which unite and bring into communication all the differential functions, i. e., the most general theory of the differential calculus."[12] Like his predecessors, says Pincherle in his *Encyclopédie*[13] article, Servois frequently confuses the expressions function and operation. This "confusion," however, was in a way implied by the nature of the subject. If we had always literally admitted the radical distinction made by Pincherle, at the beginning of his article, between the *objects* on which we operate and the *operations* executed on these objects, the ideas germinal in Leibniz's work and which we see developing in Servois's works would have been arrested in their growth. The remarkable part in the theory whose history we are briefly sketching is that the

[10] Servois, *Annales de Gergonne*, vol. V, p. 98 (1814).

[11] The *subject* of the function $f(z)$, in Servois's terminology, is z.

[12] Servois, *loc. cit.*, p. 120.

[13] Pincherle, *Encyclopédie*, etc. Vol. II, 5 fasc. I.

laws valid for a function (an object) are formally valid for an operation.[14] The distributive and commutative properties are applied by Servois to functions; his error—if we may use so strong an expression when dealing with a conception inevitable to the state of science at the time it was formulated—consisted in omitting to present the fact that the operative rules just mentioned apply as well to quantities strictly so called or to functions, as to the symbols of the operations. Thus, the character of the development of thought whose history we are endeavoring to sketch is that we have treated *operation* as an *object,* to use Pincherle's expression. The ideas found in germ both in Leibniz and in Servois have been developed by a series of mathematicians including Boole and Grassmann.

Note that Boole sets as an object for mathematics the study of operations considered in themselves independently of the various objects to which they may be applied.[15]

The formal point of view is explicitly indicated. We have sought to define the elementary notions of the calculus in view of their applications to the operations themselves. We say, for instance, that two objects are *equal* when they can be replaced by each other in a given combination, without the result being changed thereby. Now, this definition applies to symbols of operations as well as to numbers. We apply to the operations the notion of univocity (or uniformity): the operation is univocal (or uniform) if it gives one single result. Then, as stated above, we study the *operations* from the point of view of distributivity, commutativity, associativity.

The *sum* and the *product* of two operations are also defined. The sum of two operations A and B is an operation which, applied to any object, gives as a result the sum of the results of the application of the operations A and B to the object in question. We also know that the *product* of two operations A and B consists in applying to the result of the operation B the operation A. To sum up, the symbols of operations may be treated as quantities and we may subject these symbols to the other operations.

So far we have obtained a first degree of generalization, in the sense that we have presented certain general characters, such as the commutativity and the distributivity of the elementary operations of arithmetic or of the differential calculus. We ought to have gone further, however, and endeavored to expand the notion of

[14] Lagrange has already treated the symbol of differentiation as a fictitious dimension.

[15] Boole, *The Mathematical Analysis of Logic,* Cambridge, 1847, p. 3.

operation, to regard it is a correspondence, a functional relation. This has been done by Pincherle[16] and Bourlet,[17] independently of each other. The conjunction of ideas we seek to present is well characterized in Bourlet's memorandum entitled: *"Sur les opérations en général, et les équations différentielles linéaires d'ordre infini."*

'It was by seeking for the properties of the operation of derivation," writes the author at the beginning of his memorandum, "that I came to undertake the following work." Derivation comes under a very general class of operations, specially studied by Bourlet,[16] and which he calls *transmutation*: we say we have defined a transmutation when we have given a means of enabling one or more functions of the same variable to *correspond* to any function u of a regular variable x in a certain domain. The new function or functions are the transmutes of u. Change of variable, derivation, definite or indefinite integration are transmutations.

Transmutation is *univocal* (uniform) when it makes only one single transmuted function correspond to any regular function u. It is *regular* if, applied to a holomorphic function in a certain domain, it gives an equally holomorphic transmute in this domain. It is *complete,* in a certain domain, if it has a meaning for any holomorphic function in this domain. A transmutation will be designated *continuous* if the limit of the transmute of a function is the transmute of the limit of that function.

To have the *product* $T_1\ T_2$ of two transmutations T_1 and T_2, we must first take the transmute T_2 and then the transmute $T_1\ (T_2 u)$ of the result. The m^{th} *power* of a transmutation T_i is the product of m factors equal to T_1.

Speaking generally, the product of several transmutations is not commutative; thus we do not usually obtain

$$T_1\ T_2 = T_2\ T_1 .$$

The most interesting transmutations are those that verify the relation:

$$T\ (u + v = Tu + Tv .$$

Bourlet calls these transmutations *additive* transmutations. Pincherle gives the name of *distributive operations* to the operations which verify the preceding condition and the second condition:

$$Tcu = cTu .$$

[16] Pincherle, *Math. Annalen*, 1897, in which article previous notes are summed up.

[17] Bourlet, *Ann. Ecole normale*, 3, vol. XIV, 1897.

When a transmutation is continuous and regular, the second condition is reducible to the first, c being whatsoever. Hadamard gives the name of linear to Bourlet's additive transmutations. Additive transmutations are particularly important, because the change of variable $\phi(x) = f[\phi(x)]$, (f being a given function), derivation, etc., are included in the category of additive transmutations.

Every product of additive transmutations is an additive transmutation. The sum of several additive transmutations is likewise additive. Following Bourlet, we will now give the general form of an additive transmutation by supposing it universal, continuous, and regular.

Somewhat anticipating upon the following paragraphs, we will mention that Lalesco has shown that the solution of a Volterra integral equation of the first denomination, where the nucleus $N(x, s)$ is a whole analytical function in x of smaller order than I, is equivalent to the integration of a linear differential equation of infinite order with given initial conditions. This sort of equivalence between tho different analytical instruments is particularly interesting.

We must now reconsider the definition of transmutation, from which we started, to insist with Hadamard on the fact that the transmute T (u), from Bourlet's point of view, depends on the *form* of the function $u(x)$, but that it is in addition a function of the variable x. This remark will enable us to show the dependence which exists between the ideas relative to the formal development of the operative properties and another stream of ideas, that consisting in the generalization of the elementary notions of the infinitesimal calculus, with which ideas we shall now deal.

II. FUNCTIONS THAT DEPEND ON A CONTINUOUS INFINITY OF VARIABLES.

Function is the object of analysis, just as number is the object of arithmetic. The notion of whole number is simple and clear, it was known to the ancients. The notion of function is of comparatively recent date. It is a complex notion[20] the origin of which must be sought in the idea of physical law and the geometric curve. From the time of Leibniz and Bernoulli, when the word *functio* was first

[18] *Ibid.*, p. 135.

[19] Hadamard, *Leçons sur le calcul des Variations*, p. 282.

[20] P. Boutroux, *Revue de Métaphysique*, Nov., 1904.

used, the conception of function has undergone numerous transformations.

At the end of the seventeenth and in the eighteenth century, function[21] was the name generally given to a quantity y joined to a variable x by an equation including symbols that represented elementary (algebraical, trigonometrical, logarithmic) operations. Jean Bernoulli[22] in a letter addressed to Leibniz in the year 1698, seems to have been one of the first to use the term function in something approaching its modern signification. But though the word was first uttered by Bernoulli, we may affirm that, ever since Descartes, we have known something, if not of the word, at all events of the thing (curves of analytical geometry).

Under the influence of Cartesian conceptions, the curves $y=f(x)$, were considered as functions, the relation expressing true geometrical equalities. Such functions were called continuous functions (Eulerian continuity); they constituted the real functions. A function represented by several arcs of curves was considered as being formed of parts of functions. D'Alembert, Bernoulli, Euler, Lagrange, have been led to expand the meaning of the notion of function, by studying problems in physics the solution of which implied the integration of an equation with partial derivatives of the second order. Riemann, is the first part of his memorandum "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe[23]" has admirably summarized the history of these discoveries, which were destined to culminate in Fourier's fundamental work. Riemann tells of the astonishment of Lagrange, then an old man, when at the meeting of the *Académie des Sciences* on the 21st December, 1807, Fourier enunciated the proposition: that a completely arbitrary function (graphically given) could be represented by a trigonometrical series in a finite interval, and that in particular a trigonometrical development was capable of representing non-continuous functions formed of parts of functions.[24] Fourier's enunciation was not absolutely rigorous. Lajeune-Dirichlet specified the exact conditions which a function must verify to be representable by a trigonometrical series. These conditions are well known by the name of Dirichlet's

[21] Lebesgue, *Leçons sur l'ntégration*, etc., p. 2.

[22] Leibniz, *Gerhardt Math. Schriften* 3, 157.

[23] Riemann, *Werke*, p. 232.

[24] See for instance Picard, *Cours d'analyse*, vol. I, p. 244; see also Jordan, *Cours d'analyse;* Jordan, as we know, has introduced into this theory the notion of function with limited variations.

conditions.[25] Influenced by these various results, Dirichlet defined a function as follows: the quantity y is the uniform function of the quantity x in a determined interval, when to each value of x taken in the interval there corresponds one determinate value of y, without specifying at all as to the way in which the values of y depend on one another.

The conception of the function in which culminates the evolution we have just set forth is that of a correspondence in the most general meaning of the word.[26] We do not think it unnecessary to insist on the meaning of the quite arbitrary notion of function as it may subsequently be conceived in the works of D'Alembert, Fourier, Dirichlet, etc. Suppose our function is represented by a line; if this line is a true curve, the "law" which characterizes it will enable us to fix each point of it; but suppose the curve is described at hazard, that in a word the curve is given only graphically. In order that the curve thus described may be regarded as determinate, we shall here have to know all its points, since, there is no connection between the points. What method of representation shall we use for such functions? If we suppose that the function verifies Dirichlet's conditions, we know that it is possible, in a determinate interval, to represent it by a Fourier series. Consequently it is first of all necessary that the function should verify certain conditions; afterward we must note that the integrals which give the values of Fourier's coefficients can be estimated but approximately when dealing with a function given graphically.

With Dirichlet we had reached the term of generalization as regards "correspondence," but it was permitted to expand the conception we were able to form of the elements to be taken as variables. It is this generalization we shall now study.

And first of all, how have we been brought to this extension of the variable? As we have said, function is the object of analysis as number is the object of arithmetic. On function we shall be able to do operations as we do them on number. In particular we may consider function as the unknown in any relation whatsoever which we will call a *functional* relation, by reason of the nature of the unknown. In this general sense, a differential equation, an equation with partial derivatives, are functional relations, since we have to determine a function in these various cases.

In these different relations function is the unknown. Conse-

[25] Dirichlet, *Werke*, Vol. I, p. 135, and *Encyclop.*, II, I, I, p. 13.

[26] P. Boutroux, *Rev. de Métaphysique*, Nov., 1914.

quently there had necessarily to take place a *rapprochement* between the *numerical variable* and the *functional variable.*

Instead of considering a function as depending on a quantity which varies, on a variable in the ordinary use of the word, Volterra considers a function which depends on the *form of a line.* This extension certainly constitutes one of the most important events in the development of the theory of functions in our time. Is it profitable to guard the reader against the confusion that might result from Volterra's conception, with the classical notion of the function of a function? In elementary mathematics, when we say that the function f is a function of the function $\phi(x)$, which we represent by $f[\phi(x)]$ the form of f and the form of ϕ are both given: the true variable will always be the variable x on which f depends through the intermediary of ϕ.

Volterra was the first (*Rendiconti della R. Accad. dei Lincei,* Vol. III) to consider the functions of lines. We will refer to the memorandum which appeared in Vol. XII of the *Acta mathematica* (1889), a memorandum entitled: On a generalization of the theory of the functions of an imaginary variable. In a note (page 234 of the memorandum) Volterra has explained how, starting with the conception of function of lines. "In several questions of physics and analysis, we find quantities which depend *on all the values* of an ordinary function or of several totally arbitrary ordinary functions. For instance the temperature at the tip of a blade whose edge is heated depends on all the values of the temperature at the edge of the blade. Functions of lines offer another instance of like dependence."

Functions of lines enable us to depict geometrically the function which depends on all the values of a function. As an instance of the function of lines, we may consider the area comprised between the axis of the x's two ordinates raised to the points a and b and the variable portion of line connecting the two ordinates. We see that the form of the line $y=f(x)$ depends on the ordinates $y_1, y_2, \ldots y_n \ldots$ which we can raise at each point of the interval (a, b), and which form a continuous infinity. Manifestly in this sense, to consider a function which depends on the *variable* function $y=f(x)$, is to consider a function which depends on a *continuous* infinity of unknowns. The area will therefore depend on a continuous infinity of unknowns, the $y_1, y_2 \ldots$. Hadamard's *functionals* can evidently be traced to Volterra's conception. According to Hadamard

a functional is a function F which depends on the form of another function *f*. The calculus of Variations becomes, as Hadamard has remarked (*loc. cit.*, Preface) a chapter of the functional calculus.

Volterra has extended the elementary principles of the infinitesimal calculus to the new notion of function; thus we will study in succession its *continuity, derivability, variation, development in series of powers, analyticity.*

Generalization of the theory of the functions of a complex variable. We know that in the elementary theory of the functions of a complex variable, we call *monogenic* or *analytic* a function $U=P(x, y)+iQ(x, y)$, of the variable $z=x+iy$ the derivative of which is *unique* for each value of z.

The conditions of monogeneity relative to the partial derivatives of P and Q are too well known to need mention here.

Volterra has attemped to extend to three-dimensional space the theory of the functions of complex variables. To realize this extension he has had to resort to the notion of functions of lines (*Acta math.*, Vol. XII, p. 235): "We obtain generalization," he says, "by making correspond to each closed line of space the values of two imaginary variables bound to each other by a differential condition absolutely similar to the condition of *monogeneity* of the ordinary theory."

This relation similar to the monogeneity which will exist between functions of lines (imaginary variables of the quotation) will be called a relationship of *isogeneity.*

Let F and Φ be the values of two functions corresponding to a line L. Let us deform an arc A B from L and designate by Δ F and $\Delta\,\Phi$ the variations of F and of Φ. If, by indefinitely diminishing the deformation and the distance between B and the fixed point A, the relation $\Delta F/\Phi\Delta$ tends toward a limit which depends solely on the point A, we say that the two variables have a relationship of *isogeneity or* that they are *isogenous.*

Starting from this definition a theory similar to the classic theory of monogenic functions will be elaborated. Especially we shall obtain a differential relation analogous to the equation with partial derivatives of the second order already mentioned. Thus Volterra develops a general theory starting from the principles just enunciated; we shall find this theory set forth in Vol. XII of the *Acta mathematica.*

[27] Hadamard, *Leçons sur le calcul des Variations*, p. 282, etc.

[28] Volterra, *Leçons sur les équations intégrales et les équations integro-différentielles*, p. 10, etc.

We think we have adequately characterized the fundamental notions connected with the idea of function which depends on a continuous infinity of variables in an interval.

III. FUNCTIONS WHICH DEPEND ON AN ENUMERABLE INFINITY OF VARIABLES.

Whereas in the preceding paragraph we have examined the functions which depend on a *continuous* infinity of variables in an interval, we shall now consider functions which depend on a *discrete* or enumerable infinity of variables. It is in this way that Hilbert and Le Roux regard a function as a mathematical being dependent on an infinity of independent variables. For instance, Fourier's series are linear functions of their coefficients considered as independent variables. We shall see how this conception can be traced back simply to the most elementary algebraical notions. Whereas in algebra we have to determine a *finite* number of dimensions by a *finite* number of equations, in analysis we have to determine *functions* by functional equations: differential, integral equations, etc. But the second problem may be considered as a limit case of the first, if instead of considering a finite number of unknowns, determined by a finite number of equations, we consider an *infinite* number of unknowns determined by an *infinite* number of equations.

A function $\phi(x)$ being represented by a development in Fourier's series, the function may be considered as determined if we know the coefficients of Fourier's series. The unknowns $x_1, x_2 \ldots$ which constitute an enumerable infinity will be the solutions of a system of a discrete infinity of equations with a discrete infinity of unknowns. This system will be equivalent to the primitive functional relation.

Hilbert has mentioned the relations existing between his theory and the works of Hill, Poincaré, and Helge von Koch on systems of an infinity of linear equations with an infinity of unknowns and on infinite determinants. Indeed, the problem of the solution of such a system presents itself whenever we try to set up a development in series by the method of indeterminate coefficients.

Hill, in a problem of astronomy dealing with the movements of the moon, reduces the integration of a differential equation to the solution of a system of an infinite number of linear equations with an infinity of unknowns. It was natural to introduce the notion of

[29] Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig, 1912.

[30] Le Roux, *Nouvelles Annales de math.*, 1904.

determinants, as we find it in elementary algebra, into the theory of the solution of equations with an infinity of unknowns.

In his work, therefore, Hill[31] has made use of infinite determinants. Later on, Poincare[32] rigorously set up the conditions of convergence of the special infinite determinants considered by Hill. The ideas of Hill and of Poincaré have been considerably developed by Helge von Koch.

In the following section we will set forth one of the best known applications of the preceding theory, the one dealing with integral equations. It must not however be thought that that is the only application of analysis: other mathematical disciplines may be studied from the point of view of analysis with an enumerable infinity of variables: the calculus of Variations, for instance. Indeed, in the calculus of Variations, we seek to determine a function $\phi(s)$ which is to render minimum a certain dimension dependent in a given way on $\phi(s)$. If we represent the function $\phi(s)$ by means of a complete orthogonal system of functions $\phi_1(s), \phi_2(s) \ldots$ by the expression:

$$\phi(s) = x_1 \phi_1(s) + x_2 \phi_2(s) + \ldots$$

we see that our problem consists in determining the variables in infinite number $x_1, x_2 \ldots$ in such a way that a certain given function of these variables becomes minimum.

Thus the problem of the calculus of Variations is reduced to the search for a minimum, a search which belongs to the differential calculus of functions with an infinity of variables.[33] Without passing by the medium of integral equations, we may also directly broach the theory of differential equations by means of analysis with an infinity of variables.

Consequently, analysis with an infinity of variables appears less as a particular process which can be used only for a special class of problems than as a very general method which has proved its fecundity by its many applications.

IV. INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS. EQUATIONS WITH FUNCTIONAL DERIVATIVES.

In integral and integro-differential equations, as we shall see, the unknown is a function of a variable; on the other hand, in equations with functional derivatives, the unknown is a function

[31] Hill, *Acta math.*, 1888.

[32] Poincaré, *Bulletin de la Société math. de France.*

[33] Hilbert, *Rendiconti del Circolo mat. di Palermo*, vol. XXVII, 1909.

of lines. Logically, therefore, the study of these latter should be placed at the end of section II, dealing with the functions of a continuous infinity of variables (functions of lines). We have thought it preferable, however, because of the elementary character and the restricted scope of our study, to unite in one and the same paragraph everything that deals with the application of general principles to equations.

Integral equations. An integral equation is an equation in which the unknown function is represented under an integral sign. Sometimes we call such an equation a *functional equation.* Evidently we may apply the name of functional—in its broad sense—to any equation whose unknown is a function. In this way, differential equations and equations with partial derivatives would also be functional equations; speaking generally, however, following a custom which may not be altogether justified, we reserve the name of functional equations for special types of equations which are not included in one of the great classes (differential, with partial derivatives, integral, etc.), for instance, for the relations:

$$f(x+y)=f(x)+f(y)$$
$$f(x+y)=f(x).f(y)$$

considered by Legendre and Cauchy where we have to determine the unknown f; no sign of differential or integral calculus appears in these latter equations.[34]

As we know, Abel[35] was the first (in 1823 and 1826) to study, with reference to a problem in mechanics, an equation in which the unknown function is given under an integral sign. Liouville, who about the year 1832 had considered Abel's equation, showed a little later on (in 1837: *Journal de Liouville*, II, p. 24) how it was possible to obtain a particular solution of a certain linear differential equation by solving an integral equation of a somewhat different type (equation of the second kind) from that discovered by Abel (equation of the first kind). To solve the integral equation, Liouville has used a method which can be traced to the *calculus of iteration*, a calculus the principle of which may be summarily formulated as follows.[36] Let the equation be:

$$x=\mathrm{F}(x)$$

where x is an unknown number and $\mathrm{F}(x)$ a known function of x.

[34] Abel, Schröder, etc., have also considered special functional equations.
[35] Abel, *Œuvres*, I, 11.
[36] Heywood and Fréchet, *Equation de Fredholm*, etc., p. 36.

Under certain conditions of continuity we obtain the solution of this equation by indefinitely reiterating the operation F. That is to say, the succession:

$$x_1=F(x_0),\ x_2=F(x_1),\ldots.x_n=F(x_{n-1})\ldots.$$

in which x_0 is arbitrary, tends toward the root of the equation:

$$x=F(x).$$

It is possible to use a similar process for solving the integral equation found by Liouville.

The different streams of ideas, however, which we have endeavored to characterize in the two preceding sections, by going back to the principles of analysis, revealed the true meaning of integral equations and at the same time supplied the most rational methods of solution. We shall consider in succession the points of view of Volterra, Fredholm, and Hilbert. We know, in the case of the first two, that the unknown function is regarded as dependent on a continuous infinity of variables (functions dependent on all the values of a function in an interval), whereas in the case of Hilbert, the function depends on an enumerable infinity of variables. We shall try to show how, to these two fundamental conceptions, can be traced two methods of solving integral equations. Here we need not enter into the details of calculus—which after all would now be quite useless after the publication of excellent treatises dealing with the subject[37]—but simply have shown how the fundamental ideas are linked together. We shall have recourse to technical considerations only so far as is necessary for fixing the ideas.

Integro-differential equations.—The theory of the integro-differential equations now constitutes a most important chapter in analysis; there can be no doubt but that its importance will continue to increase. Not to give our work a scope of which it does not admit, we will merely offer a few general remarks on this point.

We know that in integral equations the *unknown function* figures under an integral sign. The derivatives or partial derivatives of this unknown function may also appear under the integral sign; then we find ourselves confronted with a new type of equations: the integro-differential equations studied both by Hadamard and by Volterra. All the same, the equations of Hadamard and those of Volterra are not interchangeable.

[37] Bocher, *Introduction to the Study of Integral Equations;* Lalesco, *Introduction á l'étude des équations intégrales;* and the works of Volterra, Heywood, Fréchet, etc., already mentioned.

Whereas the equations of Hadamard may be compared with ordinary differential equations, and are integrated by methods similar to those that apply to these equations, those of Volterra resemble equations with partial derivatives. As regards the applications of the integro-differential equations of Volterra, note that if the future state of a physical phenomenon depends only on its present state, or on the state infinitely near, i. e., if it is not *hereditary*, according to Picard's expression, the problems this phenomenon may raise must be solved by means of ordinary differential equations or of equations with partial derivatives. But if the future state of the phenomenon depends not only on the present state but on all the preceding states; when, in a word, it depends on the past of the phenomenon, the problems we shall have to consider will then be of the hereditary type and the equations which will suit them will then be, as a rule, integro-differential equations; the term integral representing, if we may so express it, the action of the past.

The theory of equations with functional derivatives can be traced directly to the ideas developed in our second section on Volterra's functions of lines or of surfaces and Hadamard's functionals (functions which depend on the *form* of another function). Just as a relation between a function of a variable, its ordinary derivative and the variable is called a differential equation, so a relation between a function of a line (or a functional), its derivative (in Volterra's meaning), the line and the point on which the function depends, will constitute an equation with functional derivatives.

It may happen that the function depends on two points which figure as parameters in the equation. Green's function is actually a function of two points and of a closed surface (case of space).

If the two points A and M are fixed and the surface S varies, then Green's function is a functional of S; Hadamard has calculated the infinitesimal variation of Green's function when S alone varies. Hadamard has largely developed these theories but we cannot now enter into any details. We will simply state that the equation with functional derivatives[38] which Paul Lévy has called Hadamard's equation plays an important part in these questions. Lévy remarked that this equation was *completely integrable*;[39] this latter expression having a signification analogous to that which it has in the theory of equations with total differentials.

[38] Paul Lévy, thesis, p. 21.

[39] *Ibid.*, p. 67.

V. SUGGESTIONS ON THE GENERALIZED THEORY OF ENSEMBLES.

We have seen that one of the essential characters of the functional calculus is that a function is no longer considered as solely depending on one or a finite number of variables in the ordinary sense of the word: it may depend on the form of a line, of a surface, on the form of a function, etc. In the ordinary theory of functions, we had only to consider, with reference to the variable, *ensembles* of points; we shall now have to consider *ensembles* of curves, *ensembles* of functions, etc. Consequently, the classic theory of *Ensembles,* especially the analysis of the continuum which it comprises, will have to be extended so as to apply to generalized functions, i. e., to functions of lines, to functionals, etc.

In the theory of the functions of a variable, geometric intuition supplied us with suggestions as to the nature of the linear, superficial or spatial continua which the variable was compelled to describe. As Hadamard[40] however, remarks: "The functional continuum—i. e., the multiplicity obtained by causing a function to vary continually in every possible way—presents to our mind no simple image whatsoever.

"Geometric intuition tells us nothing about it *a priori.* We are compelled to remedy this ignorance and this we can do only analytically, by creating for the use of the functional continuum a chapter on the theory of *ensembles.*"

This is the task to which Fréchet and a few other geometricians have given themselves.

Logically, this generalized theory of *ensembles* should constitute the introduction to the functional calculus, as the theory of the *ensembles* of points serves as a preface to the classic theory of the functions of a variable. In one case as in the other, however, the most abstruse logical investigations have chronologically appeared last. And as in the preceding pages we have in no way been dealing with a logical systematization of new theories—for a very good reason—we have been forced to keep to the historical sense. In Fréchet's work[41] there are two kinds of investigations closely connected; the one deals with the theory of *ensembles* considered in themselves, especially of abstract *ensembles,* i. e., *ensembles* the

[40] Hadamard, L'Enseignement mathématique, January 1, 1912.

[41] Fréchet, "Sur quelques points de calcul fonctionnel," *Rendiconti del Circolo mat. de Palermo,* Vol. XII, 1906.

nature of whose elements is not specified, the other has for its object to determine the properties of the functional operations (the most general correspondences between elements of abstract *ensembles,*[42] which are defined in these *ensembles.* Here we find notions similar to those studied in section I, on the properties of operations.

We have first to fix the principles of the theory of abstract *ensembles.* Afterward we will apply general theorems to *ensembles* of a particular nature.

The most important results of the theory of *ensembles* are those we deduce from the notion of the limit of a succession of elements. Now, "if we examine[43] the various definitions of the *limit* of a succession of numbers, or of a succession of points, or of a succession of functions, etc., we note that these definitions satisfy two conditions which may be enunciated independently of the nature of the elements considered." For the future we shall consider only *ensembles* taken from a class (L) of elements whatsoever, but satisfying the following conditions. We may recognize if two elements of a class (L) are distinct or not. We may give a definition of a succession of elements of class (L). The two conditions satisfied by the definitions of the limits are as follows:

i. If each of the elements of the infinite succession $A_1, A_2 \ldots A_n \ldots$ is identical to one and the same element A, the succession has certainly a limit which is A.

ii. If an infinite succession $A_1, A_2 \ldots A_n \ldots$ has a limit A, any succession of elements of the first succession taken in the same order: $A_{n_1}, A_{n_2}, \ldots A_{n_p} \ldots$ (the integers $n_1, n_2, \ldots n_p \ldots$ continuing to increase) has a limit which is also A.

We have now to apply to abstract *ensembles* the definitions of the theory of punctual *ensembles.*

The notions of *limit element, derived ensemble, closed ensemble, perfect ensemble,* easily extend to abstract *ensembles.* Fréchet introduces a new notion, that of *compact ensemble.* An *ensemble* is compact when it includes only a finite number of elements or when any infinity of its elements gives place at least to a limit element. When an ensemble is both compact and closed, we will call it an *extremal ensemble.* The role of the extremal ensemble in the theory of abstract *ensembles* is similar to that of the interval in the theory of linear ensembles.

[42] Montel; *Ann. Ecle normale,* 1907.

[43] Fréchet, *loc cit.,* p. 5.

We shall say that a *functional operation* V uniform in an *ensemble* E of elements of a class (L) is continuous in E, if, whatever be the element A of E limit of a succession of elements A_1, $A_2 \ldots A_n \ldots$ of E, we always have:

$$V(A) = \lim_{n=\infty} V(A_n)$$

A sequence of uniform operations in E converges *uniformly* in E toward an operation U, if, whatever be the number $\epsilon > O$, we can find an integer p such that $n < p$ has for its result $|U_n(A) - U(A)| < \epsilon$ in every element A of E.

We shall say that uniform operations[44] in one and the same *ensemble* E formed of elements of a class (L) constitute a family F of operations *equally continuous* in A in E, if, given a number $\epsilon > o$ and a succession of elements of E: A_1, $A_2 \ldots$ having for limit an element A of E, we can find an integer p so that the inequality $n > p$ has for its result:

$$|U(A) - U(A_n)| < \epsilon$$

whatever be the operation U of the family F.

The generality of the classes (L), however, does not permit of a great number of the properties of linear *ensembles* being extended to them. To obtain new properties the conception of the classes (L) must be restricted. Fréchet proceeds as follows: he considers a class (V) of elements whatsoever, though such that we can distinguish whether or not two of them are identical, and such, also, that to any two of them whatsoever A, B, we can make to correspond a number $(A,B) = (B,A) \geq o$ which has the two following properties: (i) the necessary and adequate condition that (A,B) should be nil is that A and B should be identical; (ii) whatever be the elements A, B, C, it suffices that (A,B) and (B,C) be small for the same thing to be affirmed of (A,C). The number (A,B) will be called the *vicinity* of A and of B.

The *ensemble* derived from an *ensemble* of elements of a class (V) is a closed *ensemble* [this was not necessarily the case in the classes (L)].

We shall again restrict the conception of the classes (V) but without yet specifying the nature of the elements considered.

We shall say that a succession of elements A_1, $A_2 \ldots$ of a class (V) satisfies Cauchy's conditions, when to any number $\epsilon > o$, we can

[44] Fréchet, *loc. cit.*, p. 11.

make correspond an integer n such that the inequality $(A_n, A_{n+p}) < \epsilon$ be verified what p may be.

A class (V) admits of a generalization of Cauchy's theorem, if any succession of elements of this class, which satisfies Cauchy's conditions, has a limit element.

When a class may be considered, in a way at least, as the derived *ensemble* of an enumerable *ensemble* of its own elements, it is called *separable.* Perfect and separable classes and those that admit of a generalization of Cauchy's theorem are called *normal* classes (V). Fréchet again defines the continuity of an operation by means of vicinity. In conclusion we will give only his definition of the *divergence*[45] *which* intervenes when we consider concrete examples. "It will be recognized that, in each case, it is possible to make correspond to any couple of elements A,B a number $(A,B) \geq o$ which we will call the *divergence of the two elements* and which possesses the two following properties: (i) the divergence (A,B) is nil only if A and B are identical; (ii) if A, B, C are three elements whatsoever, we always have $(A,B) \leq (A,C) + (C,B)$." The divergence satisfies the conditions imposed on the definition of vicinity.

The abstract theory on which we have just given a few suggestions, will be applied to *ensembles* whose nature will be specified. Thus Fréchet applies his general results at first to the *ensembles* of continuous functions and to the operations which bear on these functions, i. e., to the functionals; afterward he applies them to *ensembles* of points of space with an enumerable infinity of dimensions and to functions with an enumerable infinity of variables, etc. Need we remark that generalizations, like that of Fréchet, are interesting in proportion as they admit of the advance of the particular theories and the clearly determined problems from which they have sprung? By developing on their own account purely abstract theories, we should risk being led astray in the realms of formal logic and scholasticism.

In the preceding study, we hope we have set forth the importance of a line of thought which has proved itself alike fruitful in the realm of analysis, of which it is nothing but a generalization, and in that of applications. We shall not risk comparing this phase of the development of mathematics with another phase of its evolution, but we do not think we shall lay ourselves open to contradiction if we say that it appears as one of the most fruitful. It

[45] As regards Fréchet's definition of divergence Baire has made a reservation which is deserving of mention (See *Encyclopédie,* I, 1, 4, p. 530).

was no part of our plan to set forth the results which mathematical physics has obtained by the use of the new analytical instrument. Now, mathematical physics which thoroughly examines and co-ordinates the observations of experimental physics seems more and more as though it should take the place of what was formerly called, somewhat vaguely, the *philosophy of nature*. Unlike the theories, however, included under this name, mathematical physics is subject to geometrical discipline: it must prove what it advances. Moreover, it frequently contributes directions to the experimenter in his investigations. In spite of this fact, we cannot deny that mathematical physics retains an essentially theoretical and philosophical character: its main purpose is not to further the progress of industry but rather to make known the infrastructure of the universe. The functional calculus, by renewing the methods of mathematical physics, has enabled us to make further progress in the study of natural phenomena.

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BOOK REVIEWS AND NOTES.

A First Course in Nomography. By *S. Brodetsky*. London: G. Bell & Sons, Ltd., 1921. Pp. x, 135. Price, 10s. net.

Nomography owes its name, and to a large extent its development to d'Ocagne; and as a science it is best studied in d'Ocagne's own work, *Traité de Nomographie* (G. Villars, Paris, 1899). There are, however, numerous classes of students, to whom the methods of nomography are of great service; whereas, at any rate in their eyes, the science that underlies the subject is of secondary importance only. The great advantages possessed by nomograms was brought into prominence during the war; and in recent engineering treatises it is quite a customary matter to find one or more nomograms, more especially in connection with empirical formulae. For these reasons, I consider that Dr. Brodetsky, in keeping before his eyes the needs of such students, and the fact that any mathematical knowledge they have will be of the "practical" order, has made a very wise choice in his method of presentment. He states in his preface that this choice was based on experience gained in making nomograms for various technological departments in the University, and in other ways. Hence, at the very least, the elementary and clear treatment given in the book under review has the merit that it is based on real practical experience.

It certainly would have been helpful to many readers if the author had inserted, on p. 11, a geometrical figure and proof of the method for finding the ordinate of a point dividing the line joining two given points in a given ratio; but otherwise the work runs very clearly through the addition formulae, such as $la+mb$, with equally divided scales, to product formulae, such as $a^l b^m$, with logarithmic scales, and the generalisations of each. Also, it might have been wise to have compared the use of nomograms with the use of the slide-rule, and the analogy between reference lines and the cursor in simple cases; but, of course, this is not essential.

In Chapter V, the theory of parallel co-ordinates is given; and in Chapter VI, this is applied to such transcendant formulae as $nt = \phi - e \sin\phi$, and various trigonometrical formulae and equations. Chapter VII gives examples of the use of intersecting scales; and Chapter VIII gives details of empirically constructed scales, such as for the solution of $bx = a^x$. This closes a highly interesting and eminently practicable introduction to nomography, which should find a large public.

J. M. CHILDS.

ELEMENTARY VECTOR ANALYSIS. By *C. E. Weatherburn*. London: G. Bell & Sons, Ltd., 1921. Pp. xxviii, 184. Price, 12s. net.

We welcome this elementary treatise on Vector Analysis as a book that has been wanted in English for some time. The author confines himself entirely to the elementary parts of the subject, but treats them very fully. Many applications of vectors to geometry, kinematics, to dynamics, both of rigid bodies and of a particle, and to statics are given. To get the student of dynamics to think in terms of vectors is wholly to the good; for, apart from the greater elegance of vector methods, it gives him a much clearer realization of the physical realities of the problems than is attained by the ordinary analytical methods.

The more advanced parts of the subject dealing with "curls," "gradients" and "divergence," which are of such fundamental importance in the study of Electromagnetism, are omitted. It is to be hoped that the author will supplement the present work by another volume dealing with these more advanced portions, which will be as valuable to students of Electromagnetism as the present volume is to students of dynamics for whom these higher portions are unnecessary.

A word is perhaps necessary on the notation adopted for the scalar and vector product of two vectors. The author discards the familiar brackets ( ) and [ ] for these products and uses a dot and a cross respectively between the terms. One must admit that in actual work this notation is more convenient, as the brackets are then available for ordinary algebraical needs; but some uniformity of practice among writers is urgently to be desired. The subject is quite sufficiently developed now to insist upon the adoption of some standard notation for it.

ALBERT EAGLE.

AN ELEMENTARY TREATISE ON DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS. By *H. T. H. Piaggio*. London: G. Bell and Sons. Pp., xvi, 216, xxv. Price, 12s. net.

There are many physicists and engineers today, who have often longed for a short and readable treatise on Differential Equations; but who have found themselves repelled by the ponderous and academic volumes that have been produced hitherto upon the subject. Dfferential Equations, as it were, have up till now existed but for the mathematician, and have been jealously guarded by the exhaustive nature of the treatises published. Prof. Piaggio has supplied this long-felt want. In his work we have, condensed into approximately two hundred pages, an account of Differential Equations, suitable for those with no previous knowledge of the subject, and yet amply sufficient to cover the course required for an Honours degree. One feature of the book which especially appeals to us is the superabundance of excellent examples, to which are added explanations of their physical significance. The engineer will be as delighted to find how soon he can understand the phenomenon of resonance (p. 36), as will the physicist to find himself (on p. 48) solving the differential equations used by Sir J. J. Thompson in a recent important contribution to the *Philosophical Magazine*. The miscellaneous examples at the end of the book with their interesting notes—as, for example, on the Zeeman effect, on Hamilton's dynamical equations, or on Einstein's equations of Planetary Motion —will be especially welcome. Graphical methods and Numerical Approximations form an important feature of the book and are quite up-to-date, including methods hitherto unpublished. In conclusion, we congratulate the author on producing a book which is concise without being obscure, and scholarly without being academic.

J. E. J.

THE ORGANIZATION OF THOUGHT, EDUCATIONAL AND SCIENTIFIC. By *A. N.* Whitehead. Pp. viii, 228. London: Williams and Norgate. Price 6s net.

This book consists of reprints of seven addresses and other papers together with one article which has not been published before. The first five chapters deal with education, and are: "The Aims of Education—a Plea for Reform," Presidential Address to the Mathematical Association in 1916 (pp. 1-28); "Technical Education and Its Relation to Science and Literature," Presidential Address to the same Association in 1917 (pp. 29-57); "A Polytechnic in War-

time," an Address given in 1917 (pp. 58-68); "The Mathematical Curriculum," Presidential Address to the Mathematical Association in 1912 (pp. 69-91); and "The Principles of Mathematics in Relation to Elementary Teaching," a paper read to the International Congress of Mathematicians at Cambridge in 1912 (pp. 92-104). The remaining three chapters contain discussions on certain points arising in the philosophy of science, and are: "The Organization of Thought," Presidential Address to Section A of the British Association in 1916 (pp. 105-133); "The Anatomy of Some Scientific Ideas" (pp. 134-190); and "Space, Time, and Relativity," a paper read to Section A of the British Association in 1915 and afterwards to the Aristotelian Society (pp. 191-228). Of these chapters the second, sixth, and eighth have been already noticed at some length in the MONIST; thus we will here confine our attention to the remaining five essays. It must be remembered a common line of reflection runs through the whole book, and the two sections influence each other (p. v).

The two commandments to be obeyed in any educational scheme are: Do not teach too many subjects, and what you teach, teach thoroughly (p. 3). In training a child to activity of thought, we must beware of ideas that are merely received into the mind without being utilized or tested or thrown into fresh combinations (p. 4). Such "inert ideas' have overladen schools of learning at certain periods, and the above commandments indicate the way to guard against the "mental dry rot" that these inert ideas produce (p. 5). "The result of teaching small parts of a large number of subjects is the passive reception of disconnected ideas, not illumined with any spark of vitality. Let the main ideas which are introduced into a child's education be few and important, and let them be thrown into every combination possible. The child should make them his own, and should understand their application here and now in the circumstances of his actual life. From the very beginning of his education, the child should experience the joy of discovery. The discovery which he has to make is that general ideas give an understanding of that stream of events which pours through his life, which is his life. By understanding I mean more than a mere logical analysis, though that is included. I mean 'understanding' in the sense in which it is used in the French proverb, 'To understand all is to forgive all'" (pp. 5-6). The old notion that the mind is an instrument to be first sharpened and then used is "one of the most fatal, erroneous, and dangerous conceptions ever introduced into the theory of education. The mind is never passive, it is a perpetual activity, delicate, receptive, responsive to stimulus. You cannot postpone its life until you have sharpened it. Whatever interest attaches to your subject-matter must be evoked here and now; . . ." (p. 12). "There is only one subject-matter for education, and that is life in all its manifestations. Instead of this single unity, we offer children—algebra, from which nothing follows; geometry, from which nothing follows; science, from which

nothing follows; history, from which nothing follows; a couple of languages, never mastered; and lastly, most dreary of all, literature, represented by plays of Shakespeare, with philological notes and short analyses of plot and character to be in substance committed to memory" (pp. 13-14). We ought, for example, to use as examples for graphs in algebra the curves occurring in history and other studies of society (p. 16). "English education in its present phase suffers from a lack of definite aim and from an external machinery which kills its vitality. . . . England halts between two opinions. It has not decided whether to produce amateurs or experts" (pp. 25-26). "The essence of education is that it be religious. . . . A religious education is an education which inculcates duty and reverence. Duty arises from our potential control over the course of events. Where attainable knowledge could have changed the issue, ignorance has the guilt of vice. And the foundation of reverence is this perception, that the present holds within itself the complete sum of existence, backwards and forwards, that whole amplitude of time which is eternity" (p. 28).

In this address is manifested Dr. Whitehead's intense sympathy with that side of teaching in which the technical uses of an abstract science are emphasized; this by no means separates him from the majority of other mathematical logicians. To many, indeed, it seems rather striking that some of the greatest advocates of such "practical" methods of teaching mathematics are to be found among logicians. But what does, it seems, separate Dr. Whitehead from perhaps the majority of the modern school of mathematical logicians is the fact that he emphasizes the great value of the history of mathematics in teaching But the remark in the fourth chapter should be taken to heart: "Another way in which the student's ideas can be generalized is by the use of the history of mathematics, conceived not as a mere assemblage of the dates and names of men, but as an exposition of the general current of thought which occasioned the subjects to be objects of interest at the time of their first elaboration" (p. 81). "Perhaps," he says (p. 82), "it is the very subject which may best obtain the results for which I am pleading."

The fourth chapter is concerned with two sides of mathematical study (general ideas of (1) quantity and number and (2) laws of nature), and also logical method (pp. 78-79, 79-81, 82). Then there is an indication (pp. 83-89) of some stages in an ideal course of geometry "and ideals can never be realized" (p. 89).

The fifth chapter of the book is concerned with the investigation of the place which should be occupied by modern investigations respecting mathematical principles in the education of the majority of boys in secondary schools, and Dr. Whitehead's thesis is (p. 99) that a power of concentrated logical

thought "cannot be assumed, and has got to be acquired, and that a mathematical training is nothing else than the process of acquiring it." In this essay, among the memorable sentences which occur so frequently in the book the following strikes the reviewer as particularly important as bringing out that logic is really being taught even in a mathematical course. "When logic begins, definite particular individual things have been banished. I cannot relate logically this thing to that thing, for example this pen to that pen, except by the indirect way of relating some general idea which applies to this pen to some general idea which applies to that pen. And the individualities of the two pens are quite irrelevant to the logical process. . . . Thus the practice of logic is a certain way of employing the mind in the consideration of such ideas; and an elementary mathematical training is in fact nothing else but the logical use of the general ideas respecting geometry and algebra which we have enumerated above" (p. 97).

The seventh essay is not concerned with what scientists in the past meant to achieve, or thought that they could achieve. What is discussed is the natural history of ideas and not volitions of scientists. "There is a twofold scientific aim: (1) the production of theory which agrees with experience; and (2) the explanation of commonsense concepts of nature, at least in their main outlines" (p. 140). In succession the essay deals with facts, objects, time and space, and fields of force. In the third part the relation of whole and part is dealt with in a way that forms a complement to the sixth chapter, and then Dr. Whitehead's well-known construction of a mathematical "point," straight lines and planes, and empty space are shortly dealt with. In the conclusion, ontological ideas are introduced again—they were banished at the beginning of the chapter.

This book, with its emphasis on the fact that mathematics should appear to the student as a living and growing thing, and its wide sympathy with the process of becoming acquainted with abstract things by familiarity with practical applications, is the most welcome and stimulating news that has been received by learners for many years. It is to be hoped that teachers will give due weight to Dr. Whitehead's important contributions. One thing might be added. The way that suggests itself before everything else for arousing interest in a student's mind at the very beginning of work on a mathematical subject is the study of the physical and other practical problems by which the methods were developed. We can imagine no better training for anyone who is trying to get familiar with the calculus than working through the problems devised by Leibnitz and the Bernoullis toward the end of the seventeenth century. Only by such means, it seems, can one get at once an idea of what people who teach presumably mean when they talk about the "great power of the calculus" as compared with Cartesian methods. ϕ